# Capital supply uncertainty, cash holdings, and investment*

Julien Hugonnier† Semyon Malamud‡ Erwan Morellec§

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**Abstract**

We develop a model of real investment and cash holdings in which firms face uncertainty regarding their ability to raise funds in the capital markets and have to search for investors when raising outside capital. We provide an explicit characterization of the optimal investment, cash management, and dividend policies for a firm acting in the best interests of incumbent shareholders and show that capital market supply frictions have first-order effects on corporate behavior. We then use the model to explain a key set of stylized facts in corporate finance and to generate a number of novel testable implications relating the supply of funds in capital markets to corporate policy choices.

**Keywords**: Capital supply uncertainty; real investment; cash holdings; search theory.

**JEL Classification Numbers**: D83; G24; G31; G33; G35.

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†Swiss Finance Institute and Ecole Polytechnique Fédérale de Lausanne (EPFL). Postal: Ecole Polytechnique Fédérale de Lausanne, Extranef 212, Quartier UNIL-Dorigny, Ch-1015 Lausanne, Switzerland. E-mail: Julien.Hugonnier@epfl.ch.

‡Swiss Finance Institute and Ecole Polytechnique Fédérale de Lausanne (EPFL). Postal: Ecole Polytechnique Fédérale de Lausanne, Extranef 213, Quartier UNIL-Dorigny, Ch-1015 Lausanne, Switzerland. E-mail: semyon.malamud@epfl.ch.

§Corresponding author. Swiss Finance Institute, Ecole Polytechnique Fédérale de Lausanne (EPFL), and CEPR. Postal: Ecole Polytechnique Fédérale de Lausanne, Extranef 210, Quartier UNIL-Dorigny, Ch-1015 Lausanne, Switzerland. E-mail: erwan.morellec@epfl.ch.

# 1 Introduction

Following the seminal contribution of Modigliani and Miller (1958), standard valuation models in corporate finance implicitly assume that capital markets are frictionless so that firms are always able to secure funding for positive net present value (NPV) projects and cash holdings inside the firm are irrelevant. This traditional view has recently been called into question by a large number of empirical studies.[1] These studies document that firms often face uncertainty regarding their future access to capital markets and that this uncertainty has important feedback effects on corporate decisions. They also reveal that, because of the resulting liquidity risk, firms have started accumulating enormous piles of cash over the past decades, with an average cash-to-assets ratio for U.S. industrial firms that has increased from 10.5% in 1980 to 23.2% in 2006 (see e.g. Bates, Kahle, and Stulz, 2009).

While it may be clear to most economists that capital market frictions can affect corporate policy choices, it is much less clear exactly how they do so. In this paper, we develop a dynamic model of dividend, financing, and investment policies in which the Modigliani and Miller assumption of infinitely elastic supply of capital is relaxed and firms have to search for investors when raising outside funds. With this model, we seek to understand whether and when capital supply uncertainty affects corporate investment. We are also interested in determining the effects of capital markets frictions on corporate financing and dividend policies, i.e. on the firm's decision to pay out or retain earnings as well as the firm's decision to issue new securities. By answering these questions, our study aims at understanding whether the supply of capital corresponds to a separate channel through which market imperfections affect corporate behavior.

In order to aid in the intuition of the model, consider two institutional settings in which capital supply uncertainty and search frictions are likely to be especially important:

1. **Public equity offerings**: Firms can sell stock to the public either through an initial public offering (IPO), when the firm first goes public, or through seasoned equity offerings throughout

[1] See for example, Kashyap, Stein and Wilcox (1993), Kashyap, Lamont, and Stein (1994), Gan (2007), Becker (2007), Lemmon and Roberts (2007), Massa, Yasuda, and Zhang (2008), Ivashina and Scharfstein (2010), Duchin, Ozbas, and Sensoy (2010), Almeida, Campello, Laranjeira, and Weisbenner (2010), Campello, Graham, and Harvey (2010), and Choi, Getmansky, Henderson, and Tookes (2010).

the life of the firm.[2] One of the main features of public equity offerings is the book building process, whereby the lead underwriter and senior firm management travel around the country looking for investors. The main objective of this time consuming process is to search for investors until it is unlikely that the issue will fail. Yet, the risk of failure is often not eliminated and a number of IPOs are withdrawn or canceled every year. For example, Busaba, Benveniste, and Guo (2001) show that between the mid-1980s and mid-1990s almost one in five IPOs was withdrawn. Evidence from more recent periods suggests that this fraction has increased to over one in two in some years (see e.g. Dunbar and Foerster, 2008).

2. **Capital injections for private firms**: Search frictions are also important for firms that are remaining private but need new capital injections and must find investors. Indeed, while the initial capital that is required to start a business is usually provided by the entrepreneur and his family, few families have the resources to finance a growing business. When a private company decides to raise private equity capital, it must search for new investors such as angel investors, venture capital firms, or institutional investors. Even when initial outside investors are found, the firm will need to search for additional investors in every subsequent financing round (e.g. second round venture capital), facing here again a significant risk of failure.

Our analysis starts with the observation that when capital supply is uncertain, real investment and survival may depend on a firm's cash holdings. As a result, firms will choose their dividend and retention policies so as to match the future financing needs associated with these two motives, anticipating future financing constraints. To illustrate the implications of this observation for corporate policy choices, we consider a firm with assets in place that generate a continuous stream of stochastic cash flows and a real option to expand operations. The firm is financed with common equity and has the possibility to exercise its real option at any time. While standard corporate finance models assume that capital markets are frictionless, we consider instead an environment in which the firm faces uncertainty regarding its ability to raise funds in the capital markets and has to search for investors when raising outside financing. Based on these assumptions, the model yields an explicit characterization of the value-maximizing investment, dividend, and financing policies

[2] While it is commonly believed that firms rarely issue stock, Fama and French (2005) show that equity issues are commonplace with 67%, 74%, and 86% of their sample firms issuing stock every year between 1973 and 1982, between 1983 and 1992, and between 1993 and 2002, respectively.

for a firm acting in the best interests of incumbent shareholders and shows that capital supply uncertainty has first-order effects on corporate decisions.

In the model, the firm maximizes its value by making two interrelated decisions: How much cash to hold and whether to finance investment with internal or external funds. That is, the firm can retain earnings or raise outside funds and can be in one of three equity regimes: positive distributions, zero distributions, or equity issuance. If there are no frictions in the capital markets, then firms can raise as much capital as they want and there is no need to safeguard against future liquidity needs by hoarding cash. This is the traditional assumption of the theoretical literature. With capital supply uncertainty, firms find it optimal to hold cash for two motives. First, cash holdings can be used to fund profitable projects when outside funds are unavailable. Second, cash holdings can be used to cover unexpected operating losses and avoid inefficient closure. Holding cash nonetheless is costly because of the lower pecuniary return of liquid assets. Firms therefore choose their payout and retention policies to balance the benefits of cash holdings with their costs.

The analysis in the paper allows us to derive the value-maximizing level of cash holdings and to relate it to a number of firm and industry characteristics. We highlight the main empirical implications. Consistent with Opler, Pinkowitz, Stulz, and Williamson (OPSW, 1999), our model predicts that cash holdings should increase with cash flow volatility since an increase in volatility leads to an increase in the risk of inefficient closure. The model also predicts that firms with more tangible assets (i.e. firms whose assets have a higher liquidation value) should have lower cash holdings and a greater propensity to invest out of internal funds. This prediction results from to the fact that an increase in the liquidation value of assets leads to a drop in the cost of inefficient closure. Another interesting prediction of the model is that firms should always increase their cash buffer when raising funds from outside investors. This prediction is consistent with the evidence in Kim and Weisbach (2008) and McLean (2010), who find that firms' decisions to issue equity are essentially driven by their desire to build up cash reserves.

Another specific prediction of our model is that cash holdings should be used mostly to cover operating losses rather than to finance new investment, consistent with the large sample studies of OPSW (1999) and Bates, Khale and Stulz (BKS, 2009) and with the survey of Lins, Servaes, and Tufano (2010). A direct implication of this result is that cash holdings represent essentially a risk management tool aimed at insuring the firm against potential losses. In fact, we also find that when there is a large benefit to investment (when the NPV of the growth option is large) or when the opportunity cost of not investing is large (i.e. when cash flow volatility is high), it is optimal for firms to accelerate investment with internal funds by decreasing their optimal cash buffer.

The model also generates a number of predictions relating capital supply to firms' decisions. For example, we find that firms hold more cash when their access to external capital markets is more limited, in line with OPSW (1999) and BKS (2009). Another prediction of the model is that negative shocks to the supply of capital should hamper investment even if firms have enough slack to finance investment internally. Finally, and consistent with BKS (2009), our simple model produces optimal cash buffers that represent 10% to 25% of total asset value.

The present paper relates to several strands of the literature. First, it relates to the real options literature, in which it is generally assumed that firms can instantaneously tap capital markets and finance their capital expenditures by diluting equity at no cost or by issuing debt (see Dixit and Pindyck, 1994, for an early survey and Tserlukevich, 2008, Manso, 2008, Morellec and Schuerhoff, 2010, or Carlson, Fisher and Giammarino, 2010, for recent contributions). In these models, it is never optimal for firms to hold cash (whenever there is a cost of holding cash) and firms may end up raising funds infinitely many times from outside investors to cover temporary losses.

Second, our paper relates to the growing literature that examines the role of cash holdings within dynamic models. Our model is most similar to the study of Décamps and Villeneuve (DV, 2007), who examine the optimal dividend policy of a firm that has no access to external funds and owns a growth option to invest (see also Décamps, Mariotti, Rochet, and Villeneuve, 2010, or Gryglewicz, 2010, for related models). DV show that in such environments it is optimal for the firm to follow a barrier policy whereby the firm pays out dividends so as to prevent its cash buffer from exceeding an endogenously determined threshold.[3] In their analysis, DV assume that the firm can never raise outside funds and that investment is completely irreversible in that the liquidation value of assets is zero. As shown in the paper, these assumptions have important implications for firms' policy choices. Notably, we show that in the limit in which investment becomes costlessly

[3] Such policies have also recently be shown to be optimal within continuous time agency models. See for example, Biais, Mariotti, Plantin and Rochet (2007) and DeMarzo, Fishman, He, and Wang (2009).

reversible or in which access to outside capital is unrestricted, it is no longer optimal to hold cash. In addition, while the firm may decide to abandon its growth option for low levels of the cash reserves in Décamps and Villeneuve, it is never optimal to do so in our setup. More generally, one key difference between our analysis and prior contributions is that we propose and solve a dynamic model in which firms find it optimal not only to have cash holdings but also to raise funds (in discrete amounts) from outside capital markets on a regular basis, consistent with the evidence in Fama and French (2005).

Before proceeding, it should also be noted that the capital market frictions we model are in some respects similar to the financing constraints studied for example by Almeida, Campello, and Weisbach (2004) and they have some of the same implications. Models of firm behavior based on financing constraints usually predict that the agency conflicts between firm insiders and outside investors may prevent firms from raising enough capital to finance positive NPV projects and lead them to hoard cash. One important feature of these models is that they generally focus on one motive for holding cash, namely the risk of underinvestment. In addition, and more importantly, only demand factors explain variation in the firm's cash holdings, where demand factors are any firm characteristic that raises the net benefit of cash. In our model, firm behavior can only explained if one takes into account both demand (firm) and supply (market) factors. In addition, cash reserves serve two motives: financing investment and hedging negative cash flow shocks.

The remainder of the paper is organized as follows. Section 2 presents the model. Section 3 derives the optimal dividend and financing policies for a firm with no growth option when capital supply is uncertain. Section 4 allows the firm to invest in a growth option and derives the value-maximizing financing, investment, and dividend policies in this context. Section 5 develops the model's empirical predictions. Section 6 concludes. The proofs are gathered in the Appendix.

## 2 Model and assumptions

Throughout the paper, agents are risk neutral and discount cash flows at a constant rate $\rho$. Time is continuous and uncertainty is modeled by a complete probability space $(\Omega, \mathcal{F}, P; F)$, with the filtration $F = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions.

We consider a firm with assets in place and a growth option. Assets in place generate a continuous stream of cash flows $dX_t$ before investment as long as the firm is in operation. In particular, we consider that the cumulative cash flow process $X = \{X_t; t \geq 0\}$ at any time $t$ before investment is given by:

$$X_t = \int_0^t (\mu_0 ds + \sigma dB_s),$$

where $B$ is a standard $F$-Brownian motion and $(\mu_0, \sigma)$ are constant parameters representing the mean and volatility of the firm cash flows (a similar specification is used for example in Biais, Mariotti, Plantin, and Rochet, 2007, or Décamps and Villeneuve, 2007). The growth option allows the firm to increase its income stream from $dX_t$ to $dX_t + (\mu_1 - \mu_0)\, dt$, where $\mu_1 \geq \mu_0$, by paying a constant investment cost $K$. In this specification, $\mu_1 - \mu_0 \geq 0$ determines the growth potential of the firm. We consider that the firm has flexibility in the timing of investment.

Although its assets may be operated forever, the firm can also choose to abandon them. In the model, abandonment occurs either if the firm finds it optimal to liquidate its assets or if its cash buffer reaches zero following a negative shock to cash flows (i.e. if the firm is in distress). We consider that the liquidation value of assets is $\ell_i = \frac{\varphi \mu_i}{\rho}$, where $\varphi \in [0, 1]$ and $1 - \varphi$ represents a haircut related to the partial irreversibility of investment or to the costly terms that the firm has to assume in agreements for capital infusions when in distress. When $\varphi = 0$, investment is completely irreversible (or the frictions associated with cash infusions represent 100% of asset value) and the liquidation value of assets is zero. By contrast, when $\varphi = 1$, investment is costlessly reversible and there are no market frictions (and therefore no need for cash holdings). In the analysis below, we denote by $\tau_0$ the firm's stochastic liquidation time and consider that investment is at least partially irreversible in that $\varphi < 1$. This partial irreversibility may arise for example from transaction costs, from installation (and des-installation) costs, or from the firm-specific nature of capital.

We consider that management acts in the best interests of shareholders and seeks to maximize shareholder wealth when making policy choices. In the model, management selects not only the firm's investment policy but also its payout, cash management, and liquidation policies. Notably, we allow management to retain (part of) the firm's earnings inside the firm and denote by $C_t$ the amount of cash that the firm holds at any time $t$, i.e. its cash buffer. (In the following, we use indifferently the terms cash buffer, cash holdings, and cash inventory.) Cash holdings earn a constant rate of interest $r < \rho$ inside the firm and can be used to fund capital expenditures or to cover unexpected operating losses if other sources of funds are unavailable. The difference between $\rho$ and $r$ can be interpreted as a carry cost of cash.[4] As we show below, this cost implies that it is optimal for the firm to start paying dividends when its cash buffer becomes too large.

The firm can increase its cash holdings either by retaining earnings or by raising funds in the capital markets. A key difference between our setup and previous contributions is that we consider that it takes time to secure outside financing and that capital supply is uncertain. In particular, if the firm decides to increase its cash buffer or to finance the capital expenditure by raising outside funds, then it has to search for investors.[5] In the analysis below, we assume that, conditional on searching, the firm meets outside investors at the jump times of a Poisson process $N$ with constant arrival rate $\lambda > 0$. Under these assumptions, the dynamics of the firm's cash reserves are given by

$$dC_t = (rC_{t-} + \mu_0)dt + \sigma dB_t - dD_t + f_{t-}dN_t - 1_{\{t=T\}}K + 1_{\{T\leq t\}}(\mu_1 - \mu_0)dt,$$

where $T$ is the time of investment, $f$ is a nonnegative process that represents the amount of funds raised by the firm upon meeting outside investors, and $D$ is an increasing process with initial value $D_{0-} = 0$ that represents the cumulative dividends paid to shareholders. The firm's cash inventory thus grows with earnings, with outside financing, and with the interest earned on the cash inventory, and decreases with payouts to shareholders and with the cost of investment.

As documented by a series of recent empirical studies, capital supply conditions are very important in determining firms' financing decisions, the level of cash holdings, as well as the level of corporate investment (see footnote 1 above). These studies also show that firms often face uncertainty regarding their access to capital markets and that this uncertainty has important feedback effects on the firm's policy choices. Our model captures this important feature of capital markets

[4] As noted by Opler, Pinkowitz, Stulz, and Williamson (1999), the cost of holding cash includes the lower rate of return on these assets because of a liquidity premium and tax disadvantages (Graham (2000) finds that cash retentions are tax-disadvantaged because corporate tax rates generally exceed tax rates on interest income). This cost of carrying cash may also be related to a free cash flow problem within the firm.

[5] A growing body of literature argues that assets prices may be more sensitive to supply shocks than standard asset pricing theory would predict. Search theory has played a key role in the formulation of models capturing this idea (see e.g. Duffie, Garleanu, and Perdersen, 2005, Vayanos and Weill, 2008, or Lagos and Rocheteau, 2010). Duffie (2009) provides an early survey of this literature.

with the parameter $\lambda$, that governs the arrival rate of outside investors (in general, $\lambda$ may depend on both the firm's capital market access and the supply of funds in capital markets). In particular, when looking for outside funds, the probability of finding investors over each time interval $[t, t+dt]$ is $\lambda dt$ and the expected financing lag is $1/\lambda$ (years).

A comparison with some special cases to our setup illustrates how capital supply uncertainty affects firm value and corporate policy choices. When $\lambda = 0$, firms cannot raise funds in the capital markets and have to rely exclusively on internal funds to cover operating losses and to finance capital expenditures. This is the environment considered for example in Radner and Shepp (1996), Décamps and Villeneuve (2007), and Asvanunt, Broadie, and Sundaresan (2007). By contrast, when $\lambda \to \infty$, capital markets are frictionless and firms can instantly raise funds from the financial markets whenever optimal to do so. In that case, the firm has *no need for a cash buffer* and finances operating losses and capital expenditures by (costlessly) issuing new equity. This is the environment considered for example in Manso (2008), Tserlukevich (2008), Morellec and Schuerhoff (2010), or Carlson, Fisher and Giammarino (2010).[6]

With capital supply uncertainty, the problem of management is then to maximize the present value of future dividends to incumbent shareholders:

$$E_c\left[\int_0^{\tau_0} e^{-\rho t}\left(dD_t - f_{t-}dN_t\right) + e^{-\rho\tau_0}\ \left(\ell_0 + 1_{\{\tau_0 > T\}}(\ell_1 - \ell_0)\right)\right],$$

by choosing appropriately the firm's dividend, financing, and investment policies. The first term in this expression represents the present value of dividend payments to incumbent shareholders until the liquidation time $\tau_0$, net of the claim of new (outside) investors on future firm cash flows. The second term represents the present value of the cash flow to shareholders in liquidation (which depends on whether liquidation occurs before or after investment). Because management optimizes dividend policy and can always decide to pay a liquidating dividend, liquidation will occurs when the cash buffer reaches 0. As a result, management only needs to optimize over $D$ and $T$. In what follows, we denote by $V : [0, \infty) \to [0, \infty)$ the value function of this problem.

[6] In a recent paper, Morellec (2010) considers a model in which credit supply is uncertain but shareholders have deep pockets and can finance capital expenditures and operating losses when optimal to do so. In his setup firms tilt their capital structure towards equity and there is no need for cash holdings at the firm level.

## 3 Value of the firm with no growth option

To facilitate the analysis of the firm's optimization problem, we start by deriving the value of the firm when there is no growth option and the cash flow mean is $\mu_i$, denoted by $V_i(c)$. This function also gives firm value after investment if we set the cash flow mean to $\mu_1$.

When there is no growth option, the firm can follow one of three strategies: pay dividends, retain earnings and search for outside funds, or liquidate. In order to solve the firm's optimization problem, we conjecture (and later verify) that there exists a threshold $C_i^*$ for the firm's cash holdings such that the value-maximizing dividend and financing policies can be described as follows:

(a) When $c \leq C_i^*$ the firm should retain earnings, search for outside investors and increase cash holdings to the level $C_i^*$ upon finding investors;

(b) When $c > C_i^*$ the firm should distribute all cash holdings in excess of $C_i^*$.

We shall now proceed to prove this result. Since the firm's initial cash holdings can be above the threshold $C_i^*$, the value of the firm under the conjectured strategy is given by:

$$V_i(c) = c - C_i^* + V_i(C_i^*), \text{ for } c > C_i^*, \tag{1}$$

implying that it is optimal to distribute all cash holdings above $C_i^*$ with a specially designated dividend or a share repurchase. Below the threshold $C_i^*$, the optimal policy is to retain earnings and the value of the firm with no growth option satisfies the ordinary differential equation (ODE):

$$\rho V_i(c) = V_i'(c)(rc + \mu_i) + \frac{\sigma^2}{2} V_i''(c) + \lambda \left[ V_i(C_i^*) - C_i^* + c - V_i(c) \right]. \tag{2}$$

Since investors discount cash flows at the constant rate $\rho$, the left-hand side of equation (2) represents the required rate of return for investing in the firm. The right-hand side is the expected change in firm value in the region where the firm does not pay dividends. The first term captures the effects of cash savings on firm value. The second term captures the effects of cash flow volatility on firm value. The third term reflects the effects of capital supply uncertainty on firm value. This last term is the product of the probability of obtaining outside funds, given by $\lambda$, and the surplus that shareholders can extract by raising the cash buffer from its current level to its optimal level $C_i^*$, given by $V_i(C_i^*) - V_i(c) - C_i^* + c$.

The above ODE describes the dynamics of firm value when it is optimal to retain earnings and search for outside funds. When the value of the cash buffer becomes too large, it is optimal to start paying dividends. Similarly, when the cash buffer becomes too low (reaches zero), the firm is liquidated. We thus have the following boundary conditions, which characterize firm value at the liquidation and dividend thresholds:

$$V_i(0) = \ell_i, \tag{3}$$

$$\lim_{c \uparrow C_i^*} V_i(c) = V_i(C_i^*), \tag{4}$$

$$\lim_{c \uparrow C_i^*} V_i'(c) = 1, \tag{5}$$

$$\lim_{c \uparrow C_i^*} V_i''(c) = 0. \tag{6}$$

The first boundary condition reflects the fact that the liquidation value of the firm's assets is $\ell_i = \frac{\varphi \mu_i}{\rho}$ and that liquidation occurs when the cash buffer reaches zero. The second condition requires the value function in the retention region to merge with its value at the level of the cash buffer $C_i^*$ where the firm starts paying dividends. The third boundary condition reflects the fact that the firm distributes all cash holdings beyond $C_i^*$ in a minimal way, implying that the marginal value of cash holdings at that point is 1. The last condition is a high-contact condition that allows us to determine the value maximizing payout threshold $C_i^*$.

To describe the solution to the firm's problem, we need to introduce the following notation. Let

$$F_i(x) = M(-0.5\nu; 0.5; -(rx + \mu_i)^2/(\sigma^2 r)), \tag{7}$$

$$G_i(x) = \frac{rx + \mu_i}{\sigma\sqrt{r}}\, M(-0.5(\nu - 1); 1.5; -(rx + \mu_i)^2/(\sigma^2 r)), \tag{8}$$

where $\nu = (\rho + \lambda)/r$ and $M$ is the confluent hypergeometric function (see Abramowitz and Stegun, 1970, Chapter 15). Solving the firm's problem yields the following:

**Proposition 1** *There exists a unique level for the cash buffer $C_i^*$ that maximizes the value $V_i$ of a firm with no growth option, for $i = 0, 1$. This optimal cash buffer is the unique solution to*

$$\alpha_i(C_i^*)F_i(0) - \beta_i(C_i^*)G_i(0) + \frac{\lambda}{\rho + \lambda}\left(\frac{(r - \rho)C_i^* + \mu_i}{\rho} + \frac{\mu_i}{\rho + \lambda - r}\right) = \ell_i$$

*where the functions* $\alpha_i(c)$ *and* $\beta_i(c)$ *are defined by*

$$\alpha_i(c) = \frac{-G_i''(c)(\rho - r)}{2\sigma^{-3}\sqrt{r}(\rho + \lambda - r)(\rho + \lambda) e^{-(\sigma^2 r)^{-1}(rC+\mu_i)^2}},$$
$$\beta_i(c) = \frac{-F_i''(c)(\rho - r)}{2\sigma^{-3}\sqrt{r}(\rho + \lambda - r)(\rho + \lambda) e^{-(\sigma^2 r)^{-1}(rC+\mu_i)^2}}.$$

*For any level of the cash buffer* $c < C_i^*$*, the value of a firm with no growth option is*

$$V_i(c) = \alpha_i(C_i^*) F_i(c) - \beta_i(C_i^*) G_i(c) + \frac{\lambda}{\rho + \lambda}\left(V_i(C_i^*) + c - C_i^* + \frac{\mu_i + rc}{\rho + \lambda - r}\right), \tag{9}$$

*where firm value at the optimal cash buffer satisfies*

$$V_i(C_i^*) = \frac{rC_i^* + \mu_i}{\rho}.$$

The expression for the value of the firm in Proposition 1 can be interpreted as follows. The first two terms of equation (9) capture the change in firm value arising when cash holdings reach 0, at which point it is optimal to liquidate the firm's assets, or when they reach $C_i^*$, at which point it is optimal to start paying dividends to shareholders. The last term on the right hand side of equation (9) reflect the effects on firm value of the change in the cash buffer due to the arrival of outside investors. In particular, we have

$$E_c\left[e^{-\rho\theta}(V_i(C_i^*) - C_i^* + C_\theta)\right] = \frac{\lambda}{\rho + \lambda}\left(V_i(C_i^*) + c - C_i^* + \frac{\mu_i + rc}{\rho + \lambda - r}\right)$$

where $\theta$ is the (random) time at which the firm raises capital from outside investors and increases its cash holdings from their current level $c$ to the optimal level $C_i^*$.

As shown in Proposition 1, the value-maximizing level of cash holdings depends on all the parameters of the model, including the arrival rate of outside investors $\lambda$. In order to better understand the optimal strategy for the firm, Figure 1 plots $C_i^*$ as a function of the arrival rate of investors $\lambda$, the reinvestment rate $r$, the recovery rate on assets $\varphi$, and cash flow volatility $\sigma$. The base parametrization in this figure is $\rho = .06$, $r = .05$, $\lambda = 4$, $\sigma = .1$, $\mu = .1$, and $\varphi = .75$, implying a haircut of 25% of asset value in liquidation, an expected financing lag of $1/\lambda = 3$ months, and a cost of holding cash of 1% per year.

Insert Figure 1 Here

Consistent with economic intuition, the figure shows that the optimal level of cash holdings decreases with the arrival rate of investors (we prove this result in the Appendix). That is, as $\lambda$ increases, the likelihood of finding outside investors to cover unexpected operating losses increases and the need to hoard cash within the firm decreases. The figure also demonstrates that even for large values of arrival rate of investors, the firm still optimally carries a significant cash buffer. Another interesting result illustrated by the figure is that our simple model predicts a level of cash holdings that is consistent with the empirical evidence. In particular, our cash to asset ratios are between 10% and 25% (bottom panels) depending on input parameter values, as in the study of Bates, Kahle, and Stulz (2009).

Another property of the firm's optimal policy illustrated by the figure is that cash holdings should increase with cash flow volatility. This result is consistent with the evidence in Harford (1999) and Bates, Kahle, and Stulz (2009). It also suggests that the increase in cash holdings over the 1980-2005 period can be explained by the increase in idiosyncratic volatility reported by Irvine and Pontiff (2009). Our results are also consistent with the evidence in Opler, Pinkowitz, Stulz and Williamson (1999) and Bates, Kahle, and Stulz (2009), who find that firms hold more cash when their access to external capital markets is more limited. As expected, the figure also shows that cash reserves increase when the opportunity cost of holding cash decreases (i.e. when $r$ increases).

As illustrated by the figure, another key determinant of the optimal cash buffer is the liquidation value of assets (or the degree of irreversibility of investment). In particular, the optimal level of cash holdings is monotonically decreasing in $\varphi$ and converges to zero as $\varphi$ tends to one (see the Appendix). One direct testable prediction of the model is that firms with more tangible assets should have lower cash holdings. This prediction of the model is novel, and provides grounds for further empirical work on the determinants of cash holdings. Finally, in line with DeAngelo, DeAngelo, and Stulz (2006), our model predicts that the firm starts paying dividends when retained earnings reach a performance threshold.

# 4 Firm value with a growth option

We now turn to the analysis of firm value when management has the option to increase the firm's earnings by paying a lump sum cost $K$. The growth option changes the firm's policy choices and firm value only if the project has positive net present value (NPV). The following proposition provides a necessary and sufficient sufficient condition for the firm's growth option to have a positive NPV.

**Proposition 2** *The option to invest has positive net present value if and only if the cost of investment $K$ is below $K^*$ defined by:*

$$\frac{\mu_1 - \mu_0}{\rho} = K^* + \left(1 - \frac{r}{\rho}\right)(C_1^* - C_0^*),$$

*where $C_i^*$ is the value-maximizing cash inventory for a firm with assets in place that deliver a cash flow with mean $\mu_i$ and with no growth option.*

The intuition for this result is straightforward. The left hand side of equation (2) represents the expected present value of the increase in firm cash flows following the exercise of the growth option. The right hand side represents the total cost of investment, which incorporates the direct cost of investment and the change in the cost of carrying the cash balance. In the following, we consider that the cost of investment is below $K^*$ defined by equation (2) so that the value of the firm's growth option is positive.

To solve the firm's optimization problem, consider the following two alternative strategies.

(W) The firm finances investment exclusively with external funds and retains earnings until the cash buffer reaches $C_W^*$, at which point it starts paying dividends.

(U) The firm finances investment with external or internal funds, retains earnings, and invests in the growth option when the cash buffer reaches a level $C_U^*$ or upon finding investors.

Let $W(c)$ denote the value of the firm under the first strategy and $U(c)$ denote the value of the firm under the second strategy. Using standard arguments, it is immediate to show that $W(c)$ and $U(c)$ satisfy the following ODE in the continuation region where it is optimal to retain earnings:

$$\rho J(c) = J'(c)(rc + \mu_0) + \frac{\sigma^2}{2} J''(c) + \lambda \left(V_1(C_1^*) - C_1^* - K + c - J(c)\right) \tag{10}$$

for $J = W,\ U$. The left hand side of this equation is again the return required by investors for investing in the firm. The right hand side is the expected change in firm value due to the effects of a change in the cash savings (first term), cash flow volatility (second term), and the arrival of outside investors (third term). Since the firm invests in the project and readjusts its cash buffer to its optimal level after investment $C_1^*$ upon finding investors, the change in the value of incumbents shareholders due to the arrival of outside investors is given by $V_1(C_1^*) - C_1^* - K + c - J(c)$.

These ODEs are solved subject to the following boundary conditions:

$$W(0) = U(0) = \ell_0, \tag{11}$$

$$W(c) = W(C_W^*) + c - C_W^*, \text{ for } c \geq C_W^*, \tag{12}$$

$$\lim_{c \uparrow C_W^*} W'(c) = 1, \tag{13}$$

$$\lim_{c \uparrow C_W^*} W''(c) = 0, \tag{14}$$

$$U(c) = V_1(c - K), \text{ for } c \geq C_U^*, \tag{15}$$

$$U'(C_U^*) = V_1'(C_U^* - K). \tag{16}$$

These boundary conditions admit the following interpretation. Condition (11) requires firm value to be equal to $\ell_0 = \frac{\varphi \mu_0}{\rho}$ in the absence of cash reserves since the firm shuts down at $c = 0$. Condition (12) reflects the fact that it is optimal to make a payment $c - C_W^*$ to shareholders whenever cash holdings are above $C_W^*$. Condition (13) follows from condition (12) and the fact that the function $W$ is twice continuously differentiable. Condition (14) is an optimality condition that allows us to determine the value-maximizing exercise trigger $C_W^*$. Condition (15) reflects the fact that it is optimal to invest with internal funds whenever the cash buffer exceeds $C_U^*$. Finally, condition (16) is a smooth-pasting (optimality) condition that allows us to determine the value-maximizing exercise trigger $C_U^*$. Interestingly, when the cost of investment is low (i.e. for $K < \underline{K}$ defined in the Appendix), it is optimal for the firm to invest as soon as it has enough cash to pay for the capital expenditure, so that $C_U^* = K$ and condition (15) no longer applies. In that case, the firm liquidates immediately after investment.

Solving for $U(c)$ and $W(c)$ yields the following proposition.

**Proposition 3** *Assume that $K \leq K^*$ so that the growth option has positive NPV. Then the firm values associated with the strategies $(W)$ and $(U)$ are respectively given by*

$$W(c) = \begin{cases} \alpha_0(C_W^*)F_0(c) - \beta_0(C_W^*)G_0(c) + \Phi(c), & c \leq C_W^*, \\ c - C_W^* + W(C_W^*), & \textit{otherwise}, \end{cases} \tag{17}$$

*and*

$$U(c) = \begin{cases} \xi_G(C_U^*)F_0(c) - \xi_F(C_U^*)G_0(c) + \Phi(c), & c \le C_U^*, \\ V_1(c-K), & \textit{otherwise}, \end{cases} \tag{18}$$

*where*

$$\Phi(c) = \frac{\lambda(V_1(C_1^*) - (C_1^* + K))}{\rho + \lambda} + \frac{\lambda(\mu_0 + (\rho+\lambda)c)}{(\rho+\lambda)(\rho+\lambda-r)}, \tag{19}$$

*the constants* $C_W^*$ *and* $C_U^*$ *are the unique solutions to*

$$\alpha_0(C_W^*)F_0(0) - \beta_0(C_W^*)G_0(0) = \xi_G(C_U^*)F_0(0) - \xi_F(C_U^*)G_0(0) = \ell_0 - \Phi(0), \tag{20}$$

*and the functions* $\xi_F$ *and* $\xi_G$ *are defined in the Appendix.*

Having characterized firm value for the strategies (U) and (W), we now return to the firm's optimization problem. Intuitively, we expect the firm to follow strategy (U) when the cost of investment is low since in that case the cost of building up the cash buffer to the level required for investment is low, independently of the current level of cash holdings. By contrast, we expect the firm to adapt its strategy to the level of its cash holdings when the cost of investment is high. In particular, the firm should follow strategy (W) when its cash holdings are below a certain threshold $C_L^*$, since in that case it would be too costly to build up the cash buffer to invest with internal funds. Above the threshold $C_L^*$, the firm should retain earnings and invest either when its cash buffer reaches $C_H^* \ge C_U^*$ or when outside financing arrives.

Accordingly, we have that for high investment costs (i.e., for $K > K^{**}$ defined in Theorem 4 below), firm value is given by $V(c) = W(c)$ for $c \le C_L^*$, by $V(c) = V_1(c-K)$ for $c \ge C_H^*$, and satisfies the ODE:

$$\rho V(c) = V'(c)(rc + \mu_0) + \frac{\sigma^2}{2}V''(c) + \lambda\left(V_1(C_1^*) - C_1^* - K + c - V(c)\right), \tag{21}$$

between the thresholds $C_L^*$ and $C_H^*$ . This ODE is solved subject to the boundary conditions

$$V(C_L^*) = W(C_L^*), \tag{22}$$

$$V(C_H^*) = V_1(C_H^* - K), \tag{23}$$

$$V'(C_L^*) = W'(C_L^*), \tag{24}$$

$$V'(C_H^*) = V_1'(C_H^* - K),$$

which give the value of the firm at the relevant thresholds $C_L^*$ and $C_H^*$. In particular, condition (16) requires firm value to coincide with $W$ at the point $C_L^*$ where the firm switches to strategy (W). Condition (22) requires firm value to be equal to the payoff from investment when investing with internal funds at the point $C_H^* \geq C_U^*$. Finally, conditions (23) and (24) are smooth pasting conditions that allow us to determine the optimal switching points $C_L^*$ and $C_H^*$.

Solving management's optimization problem yields our main result.

**Theorem 4** *There exist two thresholds $K^{**}$ and $K^*$ for the cost of investment, with $0 < K^{**} < K^*$, such that*

(a) *If $K \geq K^*$, the growth option is worthless and optimal policy choices are as in Proposition 1.*

(b) *If $K < K^{**}$, firm value is given by $V(c) = U(c)$ and the optimal policy is to retain earnings and to invest in the growth option when the cash buffer reaches $C_U^* \leq C_1^* + K$ or when outside financing arrives.*

(c) *If $K^{**} < K < K^*$, firm value is given by*

$$V(c) = \begin{cases} W(c), & c \leq C_L^*, \\ S(c), & C_L^* \leq c \leq C_H^*, \\ V_1(c - K), & \textit{otherwise} \end{cases}$$

*where the function $S(c)$ is defined by*

$$S(c) = \xi_G(C_H^*)F_0(c) - \xi_F(C_H^*)G_0(c) + \Phi(c) \tag{25}$$

*the constants $C_L^* \geq C_W^*$ and $C_H^* \in [C_U^*, C_1^* + K]$ are the unique solutions to conditions (16) and (23). When $c \leq C_W^*$, the optimal policy is to invest exclusively with outside funds and to retain earnings until the cash buffer reaches $C_W^*$. When $c \in [C_W^*, C_L^*]$, the optimal policy is to make a lump sum payment $c - C_W^*$ and then to follow the optimal policy for $c \leq C_W^*$. When $c > C_L^*$, the optimal policy is to build up the cash buffer and exercise the option either when the cash buffer reaches $C_H^*$ or when outside financing arrives.*

(d) *If $K = K^{**}$ the firm is indifferent between (U) and (W).*

Theorem 4 provides a complete characterization of the firm's optimal policy choices and of firm value under these policies. Several important results follow from this theorem. First, the theorem shows that when $K < K^{**}$, the firm may use its cash buffer both to cover operating losses and to finance investment. Such situations arise when the cost of hoarding cash inside the firm is not too high or when the NPV of the project is large. We show below that, while cash holdings serve in principle two purposes in this case, the probability of ever financing investment with cash holdings is low, implying that cash holdings represent essentially a risk management tool aimed at insuring the firm against potential losses.

Second, Theorem 4 shows that when $K^{**} \leq K < K^{*}$, the optimal strategy for the firm depends on the current level of the cash reserves. In particular, when cash holdings are below $C_W^*$, the optimal policy is to use exclusively external funds to finance the capital expenditure and to use the cash buffer to cover operating losses. When cash holdings are between $C_W^*$ and $C_L^*$, the firm's dividend policy will consist in paying both regular dividends and specially designated dividends (see e.g. Brickley, 1983, for an economic analysis of specially designated dividends). When cash holdings are above $C_L^*$, the firm can finance the capital expenditure using either internal funds or external funds and the optimal policy is to retain earnings until the firm invests in the project.

When financing the growth option with external funds, the value-maximizing policy for the firm is to raise enough funds to finance both the capital expenditure and the potential gap between current cash holdings and the optimal level after investment $C_1^*$. By contrast, when financing the capital expenditure internally, the optimal policy is to invest at a level of cash holdings below $C_1^* + K$, implying that cash holdings are below their optimal level after investment. This effect is due to the positive discounting and to the fact that hoarding cash inside the firm is costly. Finally, since the value of the firm with a low cash balance is higher when $C_L^* > 0$ than when $C_L^* = 0$, the opportunity cost of investing for the firm is larger, implying that $C_H^* > C_U^*$.

Another interesting feature of the strategy described in Theorem 4 is that firms always increase their cash buffer when raising funds from outside investors. This prediction of the model is consistent with the evidence in Kim and Weisbach (2008) and McLean (2010), who find that firms' decisions to issue equity are essentially driven by their desire to build up cash reserves.

# 5 Model implications

## 5.1 Cash holdings before investment

When the firm has a growth option, cash holdings serve two purposes. First, they can be used to cover unexpected operating losses. Second, they can be used to finance the capital expenditure. In order to analyze the effects of this second motive on the value-maximizing level of the cash buffer, consider an economic environment in which $K < K^{**}$ so that $V(c) = U(c)$. In such an environment, the firm's cash holdings are in $(0, C_U^*)$ before investment and the difference between $C_U^*$ and $C_0^*$ represents the change in the optimal cash buffer due to the growth option.

Figure 3, Panel A, plots the optimal cash buffer $C_0^*$ and the fraction of the firm's assets accounted for by cash holdings at the optimal level of the cash buffer, defined as $\frac{C_U^*}{V(C_U^*)}$, as a function of the arrival rate of investors $\lambda$, the recovery rate on assets $\varphi$, cash flow volatility $\sigma$, and the growth potential of the firm $\mu_1 - \mu_0$. Figure 3, Panel B, plots the change in the optimal cash buffer $C_U^* - C_0^*$ due to the growth option as a function of these same parameters and as a function of the cost of investment $K$ and of the reinvestment rate $r$. In this figure, we use the same parameter values as in Figure 1 and set $K = 0.2$ and $\mu_1 = 0.125$ (so that operating cash flows increase by 25% upon investment).

Insert Figure 3 Here

Several important results follow from Figure 3. First the figure shows that the threshold $C_U^*$ exhibits the same sensitivity to the input parameters of the model than the threshold $C_0^*$. That is, the value-maximizing level of cash holdings increases with cash flow volatility $\sigma$ and the reinvestment rate $r$ and decreases with the arrival rate of investors $\lambda$ and with the recovery rate of assets in liquidation $\varphi$. (Note that $C_U^*$ converges to $K$ as $\varphi$ tends to 1 since the firm wants to be able to finance the capital expenditure with internal funds.)

Second, the figure reveals that most often the optimal level of cash holdings before investment exceeds that after investment, in that $C_U^* > C_0^*$. This is due to the fact that when the cash buffer reaches $C_U^*$, the firm needs to finance the capital expenditure out of internal funds, thereby increasing the probability of inefficient liquidation after investment. Interestingly, the figure shows that $C_U^*$ can be lower than $C_0^*$ when there is a large benefit to investment (i.e. when $\mu_1 - \mu_0$ is large) or when the risk of not investing is important (i.e. when $\sigma$ is high). That is, the optimal level of cash holdings can be lower when the firm has two motives for holding cash. This effect is mitigated by a stronger supply of capital since the risk of not being unable to cover operating losses or to finance investment is lower.

Another interesting implication of Figure 3 is that because the growth option increases firm value, cash holdings represent a smaller fraction of total asset value before investment. Here again, the level of cash holdings produced by the model are consistent with the levels reported in the recent study of Bates, Kahle, and Stulz (2009).

## 5.2 Financing investment

An important question is whether capital supply uncertainty actually affects growth and the source of financing used by firms when investing. To answer this question, consider again an economic environment in which $K < K^{**}$ so that $V(c) = U(c)$. In such economic environments, the firm invests using either internal funds or with outside capital. The probability that the firm invests using internal funds is (for $c \leq C_U^*$):

$$P_I(c) = P_c[\tau_U \leq \theta \wedge \tau_0] = E_c \left[ \int_0^\infty \lambda e^{-\lambda t} 1_{\{\tau_U \leq t \wedge \tau_0\}} dt \right] = E_c \left[ 1_{\{\tau_U \leq \tau_0\}} e^{-\lambda \tau_U} \right]$$

where $\tau_U$ is the first time that the cash reserve process

$$C_t = e^{rt} C_0 + \frac{\mu_0}{r} \left( e^{rt} - 1 \right) + \int_0^t e^{r(t-s)} \sigma dB_s$$

is equal to $C_U^* \geq 0$, and $\theta$ is the first time at which external financing becomes available. Similarly, the probability that the firm invests using external funds is

$$P_E(c) = P_c[\theta \leq \tau_U \wedge \tau_0] = 1 - P_c[\theta > \tau_U \wedge \tau_0] = 1 - E_c \left[ e^{-\lambda(\tau_U \wedge \tau_0)} \right].$$

The two probabilities do not sum to one because the firm does not invest if the cash buffer process reaches zero before investment.

Using standard arguments for one dimensional diffusion processes together with our previous results and notation, it is immediate to establish the following result:

**Proposition 5** *The probabilities that the firm invests with internal or external funds are respectively given by:*

$$P_I(c) = \alpha_I F(c) + \beta_I G(c),$$
$$P_E(c) = 1 - \alpha_E F(c) - \beta_E G(c),$$

*where the functions $F$, $G$ are defined by*

$$F(c) = F_0(c)|_{\rho=0},$$
$$G(c) = G_0(c)|_{\rho=0},$$

*with $F_0$ and $G_0$ defined by equations (7) and (8) and where the constants $\alpha_E$, $\beta_E$, $\alpha_I$, $\beta_I$ solve the following system of equations*

$$\alpha_I F(0) + \beta_I G(0) = 0,$$
$$\alpha_E F(0) + \beta_E G(0) = 1,$$
$$\alpha_j F(C_U^*) + \beta_j G(C_U^*) = 1, \text{ for } j = I, E$$

Using Proposition 5 we can examine the financing strategy of firms when investing in the growth option. In particular, Figure 4 plots the probability of investment using internal funds (red dashed line) and the probability of investment using external funds (solid blue line) as a function of the arrival rate of investors $\lambda$, the reinvestment rate $r$, the recovery rate on assets $\varphi$, cash flow volatility $\sigma$, the cost of investment $K$, and the growth potential of the firm $\mu_1 - \mu_0$. In this figure, we use the same parameter values as in Figure 1 and set the additional parameters as follows: $K = 0.2$ and $\mu_1 = 0.125$ (so that operating cash flows increase by 25% upon investment).

Consistent with economic intuition, the figure shows that as the arrival rate of investors increases, the probability of financing the capital expenditure with external funds increases. The figure also shows that the arrival rate of investors has a very important effect on these probabilities. For example, when $\lambda = 4$ (implying an expected financing lag of 3 months), the probability of investment with internal funds is 12%. When $\lambda = 12$ (implying an expected financing lag of 1 month), the probability of investment with internal funds is 3%. This result suggests that in most economic environments, cash holdings will be used mostly to cover operating losses, consistent with the evidence in the large sample studies by Opler, Pinkowitz, Stulz and Williamson (1999) and Bates, Kahle, and Stulz (2009) and in the survey of Lins, Servaes, and Tufano (2010).

Insert Figure 4 Here

Another interesting property of the model illustrated by Figure 4 is that the probability of investment with internal funds increases with the liquidation value of assets. This is a direct consequence of the fact that the investment threshold $C_U^*$ decreases with $\varphi$. Also, as volatility increases, the firm wants to hold more cash to prevent liquidation and the probability of liquidation increases. These two effects imply that the probability of investment with internal funds decreases with $\sigma$. By contrast, the probability of investment with external funds first increases (the first effect dominates) and then decreases (the second effect dominates) with $\sigma$. Finally, since $C_U^*$ increases with $K$ (see the Appendix), the probability of investment with internal funds decreases with the cost of investment.

To investigate further the effects of capital supply uncertainty on investment and liquidation probabilities, Figure 5 plots the probability of investing using internal funds, the probability of investing using external funds, and the probability of liquidation over a one-year (red dotted line) and over a three-year (solid blue line) horizon as a function of the arrival rate of investors $\lambda$. In this figure, we use the same input parameter values as in Figure 4.

Insert Figure 5 Here

Several interesting properties of the model are illustrated by Figure 5. First, the probabilities of investment with internal or external funds monotonically increase with the arrival rate of investors while the probability of liquidation monotonically decreases with this rate. Indeed, an increase in $\lambda$ results in a decrease in the investment threshold with internal funds and with an increase of the matching rate between the firm and investors. Second, and related to the above, the overall probability of investment decreases as $\lambda$ decreases, implying that a negative shock to the supply of capital may hamper investment even if firms have enough financial slack to fund all profitable investment opportunities internally. In addition, and as illustrated by the figure, the quantitative effect of a change in $\lambda$ on the probability of investment can be quite significant.

# 6 Conclusion

Following Modigliani and Miller (1958), extant theoretical research in corporate finance generally assumes that capital markets are frictionless so that corporate behavior and capital availability depend solely on firm characteristics. This demand-driven approach has recently been called into question by a large number of empirical studies. These studies document the central role of supply conditions in capital markets in explaining corporate policy choices and highlight the need for an improved understanding of the precise role of supply.

This paper takes a first step in constructing a dynamic model of corporate investment, payout, cash management, and financing decisions with capital supply effects by considering a setup in which firms face uncertainty regarding their ability to raise funds in the capital markets and have to search for investors when raising outside capital. The model provides an explicit characterization of the corporate policy choices for a firm acting in the best interests of incumbent shareholders and shows that capital market frictions have first-order effects on corporate behavior. In particular, the model shows that

1. Cash holdings should increase with cash flow volatility and decrease with the firm's access to outside capital, in line with Opler, Pinkowitz, Stulz and Williamson (1999), Harford (1999), and Bates, Kahle, and Stulz (2009).

2. Negative shocks to the supply of capital should hamper investment even if firms have enough slack to finance investment internally, consistent with the evidence in the large sample studies by Kashyap, Stein and Wilcox (1993), Gan (2007), Becker (2007), or Lemmon and Roberts (2007), and with the survey of Campello, Graham, and Harvey (2010).

3. Firms should only start paying dividends when retained earnings reach a performance threshold, in line with DeAngelo, DeAngelo, and Stulz (2006).

4. Cash holdings should be used to cover operating losses rather than to finance investment, consistent with the evidence in the large sample studies by Opler, Pinkowitz, Stulz and Williamson (1999) and Bates, Kahle, and Stulz (2009) and in the survey of corporate managers by Lins, Servaes, and Tufano (2010).

5. Firms should always increase their cash buffer when raising funds from outside investors, consistent with the evidence in Kim and Weisbach (2008) and McLean (2010).

6. Firms with better investment opportunities should find it optimal to accelerate investment with internal funds by decreasing the optimal level of cash holdings.

7. Firms with more tangible assets (with a higher liquidation value) should have lower cash holdings and should have a greater propensity to invest out of internal funds.

While some of these predictions are shared with other models, many are novel and provide grounds for further empirical work on corporate policy choices.

# Appendix

## A. Proofs of the results in Section 3

To facilitate the proofs let us start by introducing some notation that will be of repeated use throughout the appendix. Let $\mathcal{L}_i$ denote the differential operator defined by

$$\mathcal{L}_i\phi(c) := \phi'(c)(rc + \mu_i) + \frac{\sigma^2}{2}\phi''(c) - \rho\phi(c)$$

for any twice continuously differentiable function $\phi$, set

$$\mathcal{F}\phi(c) := \max_{f \geq 0} \lambda\{\phi(c + f) - \phi(c) - f\}$$

and denote by $\Theta$ the set of dividend and financing strategies such that

$$E_c\left[\int_0^{\tau_0} e^{-\rho s}(dD_s + f_{s-}dN_s)\right] < \infty$$

for all $c \geq 0$ where $\tau_0$ is the first time that the firm's cash holdings fall to zero and $E_c[\cdot]$ denotes an expectation conditional on the initial value $C_{0-} = c$.

Let $\hat{V}_i(c)$ denote the value of the firm in the absence of growth option when the mean cash flow rate is equal to $\mu_i$. In accordance with standard singular stochastic control results we have that the Hamilton-Jacobi-Bellmamn (HJB) equation is given by

$$\max\{\mathcal{L}_i\phi(c) + \mathcal{F}\phi(c), 1 - \phi'(c), \ell_i(c) - \phi(c)\} = 0 \tag{26}$$

where $\ell_i(c) := c + \varphi\mu_i/\rho$ denotes the liquidation value of the firm. Our first result shows that any classical solution to the HJB equation dominates the value of the firm.

**Lemma 6** *If $\phi$ is a twice continuously differentiable solution to (26) then $\phi(c) \geq \hat{V}_i(c)$.*

**Proof**. Let $\phi$ be as in the statement, fix a strategy $(D, f) \in \Theta$ and consider the process

$$Y_t := e^{-\rho t}\phi(C_t) + \int_{0+}^{t} e^{-\rho s}(dD_s - f_{s-}dN_s)\,.$$

Using the assumption of the statement in conjunction with Itô's formula for semimartingales (see Dellacherie and Meyer (1980, Theorem VIII–25)) we get that $dY_t = dM_t - e^{-\rho t}dA_t$ where the process $M$ is a local martingale and

$$\begin{aligned} dA_t &= (\phi(C_{t-} + f_{t-}) - \phi(C_{t-}) - f_{t-} - \mathcal{F}\phi(C_{t-}))dt \\ &\quad + (\Delta D_t - \phi(C_{t-} - \Delta D_t) + \phi(C_{t-})) + (\phi'(C_{t-}) - 1)dD_t^c. \end{aligned}$$

The definition of $\mathcal{F}$ and the fact that $\phi' \geq 1$ then imply that $A$ is nondecreasing and it follows that $Y$ is a local supermartingale. The liquidation value being nonnegative we have

$$Z_t := Y_{t\wedge\tau_0} \geq -\int_0^{\tau_0} e^{-\rho s} f_{s-} dN_s$$

and since the random variable on the right hand side is integrable by definition of the set $\Theta$ we conclude that $Z$ is a supermartingale. In particular,

$$\begin{aligned}
\phi(C_{0-}) = \phi(C_0) - \Delta\phi(C_0) &= Z_0 - \Delta\phi(C_0) \geq E_c[Z_{\tau_0}] - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau_0}\phi(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau_0}\phi(C_{\tau_0}) + \int_{0}^{\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_{0}^{\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\
&\geq E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_{0}^{\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] \qquad (27)
\end{aligned}$$

where the first inequality follows from the optional sampling theorem for supermartingales, the fifth equality follows from $C_{\tau_0} = 0$, and the last inequality follows from

$$\Delta D_0 + \Delta\phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-}-\Delta D_0}^{C_{0-}} (1 - \phi'(c))dc \leq 0$$

The desired result now follows by taking supremum over $(D, f) \in \Theta$ on both sides of (27). ■

**Lemma 7** *Let $X \geq 0$ be fixed. The unique twice continuously differentiable solution to*

$$\begin{aligned}
\mathcal{L}_i\phi(c) - \lambda(\phi(X) - X + c - \phi(c)) &= 0, \qquad c \leq X, \qquad (28)\\
\phi(c) - \phi(X) + X - c &= 0, \qquad c \geq X,
\end{aligned}$$

*is explicitly given by $\phi_i(c) = V_i(c \wedge X; X) + (c - X)^+$ where*

$$V_i(c; X) \equiv \alpha_i(X)F_i(c) - \beta_i(X)G_i(c) + \frac{\lambda}{\lambda+\rho}\left(\frac{(r-\rho)X + \mu_i}{\rho} + \frac{rc + \mu_i}{\lambda + \rho - r}\right)$$

*and the functions $\alpha_i$, $\beta_i$ are defined as in Proposition 1.*

**Lemma 8** *The general solution to the homogenous equation*

$$\lambda\phi_i(c) = \mathcal{L}_i\phi_i(c) \qquad (29)$$

*is explicitly given by*

$$\phi_i(c) = \gamma_1 F_i(c) + \gamma_2 G_i(c)$$

*for some constants $\gamma_1$, $\gamma_2$ where the functions $F_i$, $G_i$ are defined as in (7), (8).*

**Proof**. The change of variable $\phi_i(c) = g_i(-(rc + \mu_i)^2/(r\sigma^2))$ transforms equation (29) into Kummer's ODE and the conclusion now follows from standard results regarding this ODE. ■

**Lemma 9** *The functions $F_i$ and $G_i$ satisfy*

$$F_i'(c)G_i(c) - F_i(c)G_i'(c) = -\frac{e^{-(rc+\mu_i)^2/(\sigma^2 r)}}{\sigma\, r^{-1/2}}$$

*In particular, the ratio $F_i/G_i$ is monotone decreasing.*

**Proof**. The first claim follows from Abel's identity (see Hartman (1982, Section XI.2)). The second one follows because $(F_i/G_i)' = (F_i'G_i - F_iG_i')/G_i^2$. ■

**Proof of Lemma 7**. By application of Lemma 8 we have that the general solution to the second order ODE (28) is explicitly given by

$$V_i(c;X) = a_1F_i(c) + a_2G_i(c) + \frac{\lambda}{\lambda+\rho}\left(\phi_i(X) - X + \frac{(\rho+\lambda)c + \mu_i}{\lambda+\rho-r}\right)$$

for some $(a_1, a_2, \phi_i(X))$ and the proof will be complete once we show that these three unknowns are uniquely determined by the requirement that the solution be twice continuously differentiable. Using Lemma 9 in conjunction with the fact that $F_i$ and $G_i$ solve (29) we obtain that

$$\begin{aligned} -(F_i'(c)G_i''(c) - F_i''(c)G_i'(c)) &= 2\sigma^{-3}\sqrt{r}(\lambda+\rho)e^{-(rc+\mu_i)^2/(\sigma^2 r)}, \\ F_i''(c)G_i(c) - F_i(c)G_i''(c) &= 2\sigma^{-3}\sqrt{r}(rc+\mu_i)e^{-(rc+\mu_i)^2/(\sigma^2 r)}. \end{aligned} \tag{30}$$

Combining these identities with the smooth pasting and high contact conditions $V_i'(X;X) = 1$, $V_i''(X;X) = 0$ then gives $a_1 = \alpha_i(X)$, $a_2 = -\beta_i(X)$ and it now follows from (30) that

$$a_1F_i(X) + a_2G_i(X) = \frac{(\rho-r)(rX+\mu_i)}{(\lambda+\rho-r)(\lambda+\rho)}.$$

Substituting this identity into the value matching condition

$$\phi_i(X) = V_i(X;X) = a_1F_i(X) + a_2G_i(X) + \frac{\lambda}{\lambda+\rho}\left(\phi_i(X) + \frac{rX+\mu_i}{\lambda+\rho-r}\right)$$

and solving the resulting equation gives $V_i(X;X) = (rX+\mu_i)/\rho$ and completes the proof. ■

**Lemma 10** *The function $V_i(c;X)$ is increasing and concave with respect to $c \le X$, and strictly monotone decreasing with respect to $X$.*

In order to prove Lemma 10 we will rely on the following three useful results.

**Lemma 11** *Suppose that $k$ is a solution to*

$$\mathcal{L}_i k(c) + \phi(c) = 0 \tag{31}$$

*for some $\phi$. Then, $k$ does not have negative local minima if $\phi(c) \geq 0$ and $k$ does not have positive local maxima if $\phi(c) \leq 0$.*

**Proof**. At a local minimum we have $k'(c) = 0$, $k''(c) \geq 0$ and the claim follows from (31) and the nonnegativity of $\phi$. The case of a non-positive $\phi$ is analogous. ■

**Lemma 12** *Suppose that $k$ is a solution to (31) for some $\phi(c) \leq 0$ and that $k'(c_0) \leq 0$, $k(c_0) \geq 0$ and $|k(c_0)| + |k'(c_0)| + |\phi(c_0)| > 0$. Then, $k(c) > 0$ and $k'(c) < 0$ for all $c < c_0$.*

**Proof**. Suppose on the contrary that $k'(c)$ is not always negative for $c < c_0$ and let $z$ be the largest value of $c < c_0$ at which $k'(c)$ changes sign. Then, $z$ is a positive local maximum and the claim follows from Lemma 11. ■

**Lemma 13** *Suppose that $k$ is a solution to (31) for some $\phi$ such that $\phi'(c) \leq 0$ and that $k'(c_0) \geq 0$, $k''(c_0) \leq 0$ and $|k'(c_0)| + |k''(c_0)| + |\phi'(c_0)| > 0$. Then, $k'(c) > 0$ and $k''(c) < 0$ for all $c < c_0$ and $k''(c) > 0$ for $c > c_0$. In particular,*

$$k'(c_0) \;=\; \min_{c \geq 0} k'(c)\,.$$

**Proof**. Differentiating (31) shows that $m = k'$ is a solution to $\mathcal{L}_i m(c) + rm(c) + \phi'(c) = 0$ and the conclusion follows from Lemma 12. The case $c > c_0$ is analogous. ■

**Proof of Lemma 10**. As is easily seen the function

$$k(c) = V_i(c, X) - \frac{\lambda}{\lambda + \rho}\left(V_i(X; X) - X + \frac{(\rho + \lambda)c + \mu_i}{\lambda + \rho - r}\right)$$

is a solution to (29) and satisfies $k'(X) = 1 > 0$ as well as $k''(X) = 0$. In conjunction with Lemma 13 this implies that $k(c)$, and hence also $V_i(c; X)$, is increasing and concave for $c \leq X$.

To establish the required monotonicity, let $X_1 < X_2$ be fixed and consider the function $m(c) = V_i(c; X_1) - V_i(c; X_2)$. Using the first part of the proof it is easily seen that $m$ solves

$$\mathcal{L}_i m(c) - \lambda m(c) - \lambda(1 - r/\rho)(X_1 - X_2) = 0$$

with the boundary conditions $m'(X_1) = 1 - V_i'(X_1; X_2) < 0$, $m''(X_1) = -V_i''(X_1; X_2) \geq 0$. Thus it follows by a straightforward modification of Lemma 13 that $m$ is monotone decreasing and it only remains to show that $m(X_1) > 0$. To this end, observe that

$$\begin{aligned} m(X_1) &= V_i(X_1; X_1) - V_i(X_1; X_2) \\ &= V_i(X_1; X_1) - V_i(X_2; X_2) + \int_{X_2}^{X_1} V_i'(c; X_2)dc \\ &\geq V_i(X_1; X_1) - V_i(X_2; X_2) + X_2 - X_1 = (r/\rho - 1)(X_1 - X_2) > 0 \end{aligned}$$

where the first inequality follows from $V_i'(X; X) = 1$ and the first part of the proof, and the last equality follows from the fact that by assumption $\rho > r$. ■

**Lemma 14** *The unique solution to the free boundary problem (1)–(6) is given by*

$$V_i(c) = V_i(c \wedge C_i^*; C_i^*) + (c - C_i^*)^+$$

*where $C_i^*$ is the unique solution to $V_i(0, X) = \ell_i(0)$. The function $V_i$ is a twice continuously differentiable solution to (26).*

**Proof**. By Lemma 7 we have that $V_i(c)$ is twice continuously differentiable, satisfies (1) and solves (2) subject to (5), (4) and (6) so we only need to show that (3), or equivalently

$$V_i(0; C_i^*) = \ell_i(0) \tag{32}$$

uniquely determines the value $C_i^*$. By Lemma 10 we have that $V_i(0; X)$ is monotone decreasing. On the other hand, a direct calculation shows that $V_i(0; 0) = \mu_i/\rho > 0$, $V_i(0; \infty) < 0$ and it follows that (32) has a unique solution. To complete the proof it remains to show that $V_i$ is a solution to the HJB equation. Using the concavity of $V_i(c)$ in conjunction with the smooth pasting condition we obtain that $1 - V_i'(c)$ is negative below the threshold $C_i^*$ and zero otherwise so that

$$\ell_i(c) - V_i(c) = \int_0^c (1 - V_i'(x))dx \leq 0.$$

On the other hand, using the concavity of $V_i(c) = V_i(c; C_i^*)$ in conjunction with Lemma 7 and the smooth pasting condition we obtain

$$\begin{aligned} (\mathcal{L}_i + \mathcal{F})V_i(c) &= \mathcal{L}_i V_i(c) + 1_{\{c < C_i^*\}}(V_i(C_i^*) - X + c - V_i(c)) = 1_{\{c \geq C_i^*\}}\mathcal{L}_i V_i(c) \\ &= (r - \rho)(c - C_i^*)^+ < 0 \end{aligned}$$

Combining the above results shows that $V_i$ is a solution to (26) and completes the proof. ■

**Proof of Proposition 1**. Combining the results of Lemmas 6 and 14 shows that $V_i \geq \hat{V}_i$. In order to establish the reverse inequality consider the dividend and financing strategy defined by $D_t^* = L_t$ and $f_t^* = (C_i^* - C_t)^+$ where the process $C$ evolves according to

$$dC_t = (rC_{t-} + \mu_i)dt + \sigma dB_t - dD_t^* + f_{t-}^* dN_t$$

with initial condition $C_{0-} = c \geq 0$ and $L_t = \sup_{s\leq t}(X_t - C_i^*)^+$ where

$$dX_t = (rX_{t-} + \mu_i)dt + \sigma dB_t + (C_i^* - X_{t-})^+ dN_t\,.$$

In order to show that the strategy $(D^*, f^*)$ is admissible we start by observing that due to standard properties of Poisson processes we have

$$E_c\left[\int_0^\infty e^{-\rho t} f_{t-}^* dN_t\right] \leq E_c\left[\int_0^\infty e^{-\rho t} C_i^* dN_t\right] = \frac{\lambda C_i^*}{\rho}$$

where the inequality follows from the definition of $f^*$. Using this bound in conjunction with Itô's lemma and the assumption that $r < \rho$ we then obtain that

$$\begin{aligned} E_c\left[\int_0^t e^{-\rho s} dD_s^*\right] &= C_0 + E_c\left[\int_0^t e^{-\rho s}((r-\rho)C_{s-} + \mu_i)ds + \int_0^t e^{-\rho s} f_{s-}^* dN_s\right] \\ &\leq C_0 + E_c\left[\int_0^\infty e^{-\rho s}\mu_i ds + \int_0^\infty e^{-\rho s} f_{s-}^* dN_s\right] \leq C_0 + \frac{1}{\rho}(\mu_i + \lambda C_i^*) \end{aligned}$$

holds for any finite $t$ and it now follows from Fatou's lemma that $(D^*, f^*) \in \Theta$. Applying Itô's formula for semimartingales to the process

$$Y_t = e^{-\rho t\wedge\tau_0} V_i(C_{t\wedge\tau_0}) + \int_{0+}^{t\wedge\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)$$

and using the definition of $(D^*, f^*)$ in conjunction with the fact that the function $v_i$ solves the HJB equation we obtain that the process $Y$ is a local martingale. Now, using the fact that $C_t \in [0, C_i^*]$ for all $t \geq 0$ together with the increase of $V_i$ we deduce that

$$|Y_\theta| \leq |V_i(C_i^*)| + \int_0^\infty e^{-\rho t}(dD_t^* + f_{t-}^* dN_t)$$

for any stopping time $\theta$ and since the random variable on the right hand side is integrable we conclude that the process $Y$ is a uniformly integrable martingale. In particular, we have

$$\begin{aligned} V_i(c) = Y_{0-} = Y_0 - \Delta Y_0 = Y_0 + \Delta D_0^* &= E_c[Y_{\tau_0}] + \Delta D_0^* \\ &= E_c\left[e^{-\rho\tau_0} V_i(C_{\tau_0}) + \int_{0+}^{\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)\right] + \Delta D_0^* \\ &= E_c\left[e^{-\rho\tau_0}\ell_i(0) + \int_0^{\tau_0} e^{-\rho s}(dD_s^* - f_{s-}^* dN_s)\right] \end{aligned}$$

where the third equality follows from the definition of $V_i$ and the fourth one follows from the martingale property of $Y$. This shows that $V_i \geq \hat{V}_i$ and establishes the optimality of $(D^*, f^*)$. ■

**Lemma 15** *The level of cash holdings $C_i^*$ that is optimal for a firm with no growth option is monotone decreasing in $\lambda$ and $\varphi$.*

**Proof**. Monotonicity in $\varphi$ follows from the definition of $C_i^*$ and the monotonicity of $\ell_i$. To establish the required monotonicity in $\lambda$ it suffices to show that $V_i(0; X, \lambda)$ is monotone decreasing in $\lambda$. Indeed, in this case we have

$$\ell_i(0) = V_i(0; C_i^*(\lambda_1), \lambda_1) \leq V_i(0; C_i^*(\lambda_1); \lambda_2)$$

for all $\lambda_1 < \lambda_2$ and therefore $C_i^*(\lambda_2) \leq C_i^*(\lambda_1)$ due to the fact that $V_i(0; X, \lambda)$ is decreasing in $X$. To establish the required monotonicity observe that $V_i(X; X, \lambda) = \frac{rX+\mu_i}{\rho}$ does not depend on $\lambda$. As a result, it follows from Lemma 7 that the function defined by

$$k(c) = V_i(c; X, \lambda_1) - V_i(c; X, \lambda_2)$$

for some $\lambda_1 < \lambda_2$ satisfies

$$k(X) = k'(X) = k''(X) = k^{(3)}(X) = k^{(4)}(X) = 0$$

and solves the ODE

$$\mathcal{L}_i k(c) - \lambda_1 k(c) = (\lambda_2 - \lambda_1)(V_i(X; X, \lambda_2) - V_i(c; X, \lambda_2) - (X - c)). \tag{33}$$

Since, by Lemma 10, $V_i(c; X, \lambda_2)$ is concave in $c$ and $V_i'(X; X, \lambda_2) = 1$, the right hand side of (33) is nonnegative for all $c \leq X$. It follows by slight modification of Lemma 11 that $k(c)$ cannot have a positive local maximum. Since

$$k^{(5)}(X) = \frac{2}{\sigma^2}(\lambda_1 - \lambda_2)V_i^{(3)}(X; X, \lambda_2) = \frac{2}{\sigma^2}(\lambda_1 - \lambda_2)(\rho - r) < 0\,,$$

we conclude that $k$ is decreasing in a small neighborhood of $X$. Therefore, $k(c)$ is decreasing for all $c \leq X$ and hence $k(c) > k(X) = 0$ for all $c \leq X$. ■

## B. Proof of Proposition 2

The proof of Proposition 2 will be based on a series of lemmas. To facilitate the presentation of the results in this appendix and the next. Let $\hat{V}$ denote the value of the firm and $\Pi$ denote the set of triples $\pi = (\tau, D, f)$ where $\tau$ is a stopping time that represents the firm's investment time and $(D, f) \in \Theta$ is an admissible dividend and financing strategy.

**Lemma 16** *The value of the firm satisfies*

$$\hat{V}(c) = \sup_{\pi \in \Pi} E_c \left[ \int_0^{\tau \wedge \tau_0} e^{-\rho t} \left( dD_t - f_{t-} dN_t \right) + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_1(C_\tau) \right].$$

*In particular, if $V_0(c) \geq V_1(c - K)$ for all $c \geq K$ then it is optimal to abandon the growth option.*

**Proof**. The proof of the first part follows from standard dynamic programming arguments and therefore is omitted. To establish the second part assume that $V_0(c) \geq V_1(c-K)$ for all $c \geq K$ and observe that $\Delta C_\tau = -K + 1_{\{\tau \in \mathcal{N}\}} f_{\tau-}$ where $\mathcal{N}$ denotes the set of jump times of the Poisson process. Using this identity in conjunction with the first part we obtain

$$V(c) \leq \sup_{\pi \in \Pi} E_c \bigg[ \int_0^{\tau \wedge \tau_0} e^{-\rho t}(dD_t - f_{t-} dN_t) \\ + 1_{\{\tau \geq \tau_0\}} e^{-\rho \tau_0} \ell_0(0) + 1_{\{\tau < \tau_0\}} e^{-\rho \tau} V_0(C_{\tau-} + 1_{\{\tau \in \mathcal{N}\}} f_{\tau-}) \bigg].$$

and the desired result follows since the right hand side of this inequality is equal to $V_0(c)$ by standard dynamic programming arguments. ■

In order to establish Proposition 2 it now suffices to show that $V_0(c) \geq V_1(c-K)$ for all $c \geq K$ if and only if $K \geq K^*$. This is the objective of the following:

**Lemma 17** *The constant $K^*$ is nonnegative and the following are equivalent:*

(1) $K \geq K^*$,

(2) $K \geq V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*)$,

(3) $V_0(c) \geq V_1(c-K)$ *for all* $c \geq K$.

**Proof**. The equivalence of (1) and (2) follows from the definition of $K^*$ and the fact that by Proposition 1 we have

$$V_i(C_i^*) = \frac{rC_i^* + \mu_i}{\rho}.$$

In order to show that the constant $K^*$ is nonnegative we argue as follows: Since $\mu_0 < \mu_1$, the set of feasible strategies for $V_0$ is included in the set of feasible strategies for $V_1$. It follows that $V_0 \leq V_1$ and combining this with the definition of $C_i^*$ shows that

$$\begin{aligned} K^* &= V_1(C_1^*) - V_0(C_0^*) - (C_1^* - C_0^*) \\ &= \max_{C \geq 0}\{V_1(C) - C\} - \max_{C \geq 0}\{V_0(C) - C\} \geq 0. \end{aligned}$$

To establish the implication (1) $\Rightarrow$ (3) it suffices to show that under (1) we have $V_1(c-K^*) \leq V_0(c)$ for all $c \geq K^*$. Indeed, if that is the case then (3) also holds since

$$V_1(c-K) \leq V_1(c-K^*), \qquad c \geq K \geq K^*$$

due to the increase of the function $V_1$. For $c \geq K^* \vee C_0^*$ the concavity of the function $V_1$ and the fact that the function $V_0$ is linear with slope one above the level $C_0^*$ imply that

$$\begin{aligned} V_1(c - K^*) &\leq V_1(C_1^*) + (c - K^* - C_1^*) \\ &= V_0(c) + C_0^* - V_0(C_0^*) - K^* - C_1^* = V_0(c) \end{aligned}$$

so it remains to prove the result for $c \in [K^*, C_0^*]$. Consider the function $k(c) = V_0(c) - V_1(c - K^*)$. Using Lemma 7 in conjunction with the fact that $C_0^* < C_1^* + K^*$ by Lemma 18 below we have that the function $k$ is a solution to

$$\mathcal{L}_0 k(c) - \lambda k(c) + (-\mu_1 + \mu_0 + rK^*)V_1'(c - K^*) = 0$$

on the interval $[K^*, C_0^*]$. Combining Lemma 18 below with the increase of $V_1$ shows that the last term on the left hand side of this equation is positive and since

$$\begin{aligned} k(C_0^*) &= V_0(C_0^*) - V_1(C_0^* - K^*) \geq V_0(C_0^*) - V_1(C^*) - (C_0^* - K^* - C_1^*) = 0, \\ k'(C_0^*) &= V_0'(C_0^*) - V_1'(C_0^* - K^*) = 1 - V_1'(C_0^* - K^*) \geq 0 \end{aligned}$$

by the concavity of $V_1$, we can apply Lemma 12 to conclude that $k(c) \geq 0$ for all $c \leq C_0^*$. Finally, the implication (3) $\Rightarrow$ (2) follows by taking $c > C_0^* \vee (C_1^* + K)$ ■

**Lemma 18** *We have $C_0^* < C_1^* + K^*$ and $\mu_1 - \mu_0 - rK^* > 0$.*

**Proof**. The definition of the constant $K^*$ implies that the first inequality is equivalent to the second which is in turn equivalent to

$$C_1^* - C_0^* > \frac{\mu_0 - \mu_1}{r}.$$

To prove this inequality, it suffices to show that in the absence of a growth option the optimal level of cash holdings $C_i^* = C^*(\mu_i)$ satisfies

$$-\frac{\partial C^*(\mu)}{\partial \mu} < \frac{1}{r}. \tag{34}$$

By an application of Lemma 14, we have that

$$\tilde{V}(0; C^*(\mu), \mu) + \frac{\lambda}{\lambda + \rho}\left(\frac{\mu + (r - \rho)C^*(\mu)}{\rho} + \frac{\mu}{\lambda + \rho - r}\right) = \frac{\varphi\mu}{\rho}$$

where the function $\tilde{V}$ is defined by

$$\tilde{V}(0; X, \mu) = \alpha(X; \mu)F(0; \mu) - \beta(X; \mu)G(0; \mu). \tag{35}$$

Using (7), (8) in conjunction with the definition of the functions $\alpha$ and $\beta$ we obtain

$$\tilde{V}_\mu(0;X,\mu) = \frac{1}{r}\left(\tilde{V}_c(0;X,\mu) + \tilde{V}_X(0;X,\mu)\right).$$

where a subscript denotes a partial derivative and it follows that

$$\frac{\partial C^*(\mu)}{\partial \mu} = \frac{-\tilde{V}_X(0;C^*(\mu),\mu)/r - \tilde{V}_c(0;C^*(\mu);\mu)/r + \varphi/\rho - B}{\tilde{V}_X(0;C^*(\mu),\mu) - A}$$

where we have set

$$A = \frac{\lambda}{\lambda+\rho}\left(1 - \frac{r}{\rho}\right), \quad B = \frac{\lambda}{\lambda+\rho}\left(\frac{1}{\rho} + \frac{1}{\lambda+\rho-r}\right).$$

By Lemma 10 we have that the function $\tilde{V}$ is decreasing in $X$ and since $A > 0$ it follows that the validity of equation (34) is equivalent to

$$-\tilde{V}_X(0;C^*(\mu),\mu) - \tilde{V}_c(0;X;\mu) - r\left(B - \frac{\varphi}{\rho}\right) < A - \tilde{V}_X(0;C^*(\mu),\mu),$$

which in turn follows from

$$\tilde{V}_c(0;X;\mu) + r\left(B - \frac{1}{\rho}\right) > 0. \tag{36}$$

Since the difference $\tilde{V} - V$ is a linear function of $c$ we have from Lemma 10 that the function $\tilde{V}$ is concave. Thus, it follows from the smooth pasting condition that

$$\tilde{V}_c(0;C^*(\mu),\mu) = V_c(0;C^*(\mu),\mu) - \frac{\lambda}{\lambda+\rho-r} \geq V_c(C^*(\mu);C^*(\mu),\mu) - \frac{\lambda}{\lambda+\rho-r} = \frac{\rho-r}{\lambda+\rho-r}$$

and combining this inequality with a straightforward calculation shows that (36) holds. ■

## C. Proof of Proposition 3 and some additional results.

Denote the value of the firm by $\hat{V}$. Proposition 3 directly follows from the following:

**Lemma 19** *The unique smooth solution to the free boundary problem defined by* (10), (11), (12) *and* (13) *is given by* $(W, C^*_W)$ *where the function* $W$ *is defined by* (17) *and the constant* $C^*_W$ *is the unique solution to* (20). *The function* $W$ *is increasing, concave and satisfies* $W(c) \leq \hat{V}(c)$.

**Proof**. The results follow by arguments similar to those we used in the proof of Proposition 1. We omit the details. ■

The following result follows by direct calculation.

**Lemma 20** *The unique solution* $\psi(c;K)$ *to equation* (10) *satisfying* $\psi(0;K) = \ell_0,\ \psi(K;K) = \ell_1$ *is explicitly given by*

$$\psi(c;K) \;=\; a_1(K)\,F_0(c) \;-\; b_1(K)\,G_0(c) \;+\; \Phi(c,K)$$

*where the function* $a_1$ *and* $b_1$ *are defined by*

$$a_1(K) \;=\; \frac{G_0(K)(\ell_0 - \Phi(0;K)) - G_0(0)(\ell_1 - \Phi(K;K))}{G_0(K)F_0(0) - F_0(K)G_0(0)}$$

*and*

$$b_1(K) \;=\; \frac{F_0(K)(\ell_0 - \Phi(0;K)) - F_0(0)(\ell_1 - \Phi(K;K))}{G_0(K)F_0(0) - F_0(K)G_0(0)}$$

*with* $\Phi = \Phi(c;K)$ *as in equation* (19).

**Lemma 21** *Let* $\psi'(K;K) < V_1'(0)$*. Then, the unique solution to the free boundary problem defined by* (10), (11), (14), (15) *is given by* $(U, C_U^*)$ *where the function* $U$ *is defined by* (18) *and the constant* $C_U^*$ *is the unique solution to* (20) *with*

$$\xi_G(x) = e^{(rx+\mu_0)^2/(r\sigma^2)}\sigma r^{-1/2}\left(G_0'(x)v_1(x-K) - G_0(x)v_1'(x-K)\right),$$
$$\xi_F(x) = e^{(rx+\mu_0)^2/(r\sigma^2)}\sigma r^{-1/2}\left(F_0'(x)v_1(x-K) - F_0(x)v_1'(x-K)\right),$$

*and*

$$v_1(x) = V_1(x) - \Phi(x+K)\,. \tag{37}$$

*If* $\psi'(K;K) > V_1'(0)$ *then we let* $C_U^* = K$ *and the function* $U$ *is defined by* (18) *with* $\xi_G(K) = a_1(K),\ \xi_F(K) = b_1(K)$*.*

**Proof**. Using arguments similar to those of the proof of Lemma 7 it can be shown that the unique solution to (10) such that (14) and (15) hold for $c = C_U^*$ is increasing and is given by

$$U(c) = \xi_G(C_U^*)F_0(c) - \xi_F(C_U^*)G_0(c) + \Phi(c)$$

for $c \le C_U^*$. As a result, the first part of the proof will be complete once we show that the value matching condition at zero

$$U(0) = \xi_G(C_U^*)F_0(0) - \xi_F(C_U^*)G_0(0) + \Phi(0) = \ell_0 \tag{38}$$

admits a unique solution $C_U^* \le C_1^* + K$ when $\psi'(K;K) \le V_1'(0)$. To this end we start by observing that, as a result of Lemma 22 below, finding a solution to the free boundary problem (10), (11), (14), (15) is equivalent to finding a linear function $\phi$ that is tangent to the graph of the function $\hat{v}_1$ defined by

$$v_1(c-K) = F_0(c)\hat{v}_1(Z(c)) = F_0(c)\hat{v}_1\left(\frac{G_0(c)}{F_0(c)}\right) \tag{39}$$

and such that

$$\phi(Z(0))F_0(0) = \ell_0 - \Phi(0).$$

A direct calculation using the results of Lemmas 14 and 18 shows that

$$\mathcal{L}_0 v_1(c-K) - \lambda v_1(c) = (r-\rho)(c - C_1^* - K)^+ + (-\mu_1 + \mu_0 + rK)V_1'(c-K) \leq 0$$

for all $c \geq K$ and it now follows from Lemma 22 that $\hat{v}_1(y)$ is concave for all $y \geq Z(K)$. On the other hand, since $V_1$ is concave, we obtain

$$v_1'(c-K) = V_1(c-K) - \frac{\lambda}{\lambda+\rho-r} \geq V_1'(C_1^*) - \frac{\lambda}{\lambda+\rho-r} = \frac{\rho-r}{\lambda+\rho-r} > 0$$

and therefore $v_1(c-K)$ is positive for sufficiently large values of $c$. Since $F_0$ is nonnnegative and decreasing, this implies that the ratio $v_1(c-K)/F_0(c)$ is increasing for large $c$ and it now follows from Lemma 22 that $\hat{v}_1(y)$ is increasing for large values of $y$ and is therefore globally increasing in $y \geq Z(K)$ since $\hat{v}_1(y)$ is concave.

Since $\psi'(K;K) \leq V_1'(0)$, the line passing through $(Z(0), (\ell_0 - \Phi(0))/F_0(0))$ and $(Z(K), (\ell_1 - \Phi(0))/F_0(0))$ has a higher slope at $y = Z(K)$ than $\hat{v}_1$. Using the concavity and increase of $\hat{v}_1$, it is then immediate that there exists a unique line passing through $(Z(0), (\ell_0 - \Phi(0))/F_0(0))$ that is tangent to $\hat{v}_1(y)$ at some $y > Z(K)$. Setting $C_U^* = Z^{-1}(y^*)$ proves the existence of a unique solution to the value matching condition (38). Since $\hat{v}_1$ is concave, it lies below its tangent line at $y^*$ and, transforming back to $V_1(c-K)$ and $U(c)$, we get $U(c) \geq V_1(c-K)$.

In order to prove that $U(c) \leq \hat{V}(c)$ and thus complete the proof, consider the investment, dividend and financing strategy $\pi^U$ defined by $\tau = \tau_N \wedge \tau_U^*$, $D^U = 0$ and

$$f_t^U = (C_1^* + K - C_{t-})^+ \tag{40}$$

where $\tau_N$ denotes the first jump time of the Poisson process and $\tau_U^*$ denotes the first time that the firm's cash reserves reach the level $C_U^*$. As is easily seen, we have

$$E_c\left[\int_0^{\tau_0} e^{-\rho t}\left(dD_t^U + f_{t-}^U dN_t\right)\right] \leq E_c\left[\int_0^{\infty} e^{-\rho t}(C_1^* + K) dN_t\right] = \frac{\lambda}{\lambda+\rho}(C_1^* + K)$$

and it follows that $\pi^U \in \Pi$. On the other hand, using an argument similar to that of the proof of Proposition 1 it can be shown that the process

$$Y_t = e^{-\rho t \wedge \tau_0 \wedge \tau_U^*} U(C_{t \wedge \tau_0 \wedge \tau_U^*}) + \int_0^{t \wedge \tau_0 \wedge \tau_U^*} e^{-\rho t}\left(dD_s^U - f_{s-}^U dN_s\right)$$

is a uniformly integrable martingale and applying the optional sampling theorem at the stopping time $\tau_N$ implies

$$U(c) = Y_0 = E[Y_{\tau_N}] = E_c\left[e^{-\rho\tau\wedge\tau_0}U(C_{\tau\wedge\tau_0}) + \int_0^{\tau\wedge\tau_0} e^{-\rho t}\left(dD_s^U - f_{s-}^U dN_s\right)\right]$$
$$= E_c\left[1_{\{\tau<\tau_0\}}e^{-\rho\tau}V_1(C_\tau) + 1_{\{\tau_0\le\tau\}}e^{-\rho\tau_0}\ell_0 + \int_0^{\tau\wedge\tau_0} e^{-\rho t}\left(dD_s^U - f_{s-}^U dN_s\right)\right]$$

and the desired result now follows from Lemma 16 by taking the supremum over the set of admissible strategies on both sides. ∎

**Lemma 22** *Let $q$ denote an arbitrary function and define $\hat{q}$ implicitly through*

$$q(c) = F_0(c)\hat{q}(Z(c)) = F_0(c)\hat{q}\left(\frac{G_0(c)}{F_0(c)}\right).$$

*Then we have:*

(a) *The function $Z$ is monotone increasing and $\hat{q}(y) = q(Z^{-1}(y))/F_0(Z^{-1}(y))$,*

(b) *The function $q$ solves (29) if and only if the function $\hat{q}$ is linear,*

(c) *For an arbitrary $c \ge 0$,*

$$\hat{q}'(y)(q(c)/F_0(c))' \ge 0,$$
$$\hat{q}''(y)(\mathcal{L}_0 q(c) - \lambda q(c))) \ge 0,$$

*with $y = Z(c)$.*

**Proof**. The first two claims follow by direct calculation using the definition of $\hat{q}$, $F_0$ and $G_0$. The third claim is formula (6.2) in Dayanik and Karatzas (2003). ∎

**Lemma 23** *The threshold $C_W^* = C_W^*(K)$ is decreasing in $K$ and satisfies $C_W^*(K^*) = C_0^*$.*

**Proof**. By (20), we have that $C_W^*$ is the unique solution $X$ to

$$\ell_0(0) = \tilde{V}_0(0; X) + \frac{\lambda}{\lambda+\rho}\left((r/\rho - 1)\, C_1^* - K + \frac{\mu_1}{\rho} + \frac{\mu_0}{\lambda+\rho-r}\right)$$

where $\tilde{V}$ is defined in (35). By the proof of Lemma 10, $\tilde{V}_0(0; X)$ is monotone decreasing in $X$ and the desired monotonicity with respect to $K$ follows by differentiation. To show that $C_W^*$ converges to $C_0^*$ as $K \to K^*$ we argue as follows. By definition of $K^*$ we have

$$V_1(C_1^*) - C_1^* - K^* = V_0(C_0^*) - C_0^*.$$

Thus, it follows from Lemma 14 that the function $V_0$ solves

$$0 = \mathcal{L}_0 V_0(c) + \lambda[V_0(C_0^*) - C_0^* + c - V_0(c)] = \mathcal{L}_0 V_0(c) + \lambda[V_1(C_1^*) - C_1^* - K^* + c - V_0(c)]$$

on the interval $[0, C_0^*]$ with the boundary conditions $V_0'(C_0^*) = 1$, $V''(C_0^*) = 0$ and the desired result follows from the uniqueness part of Lemma 19. ■

**Lemma 24** *The following are equivalent:*

(1) $K > K^*$,

(2) $W(C_W^*(K)) - C_W^*(K) < V_1(C_1^*) - (C_1^* + K)$.

**Proof**. Evaluating the ODE

$$\mathcal{L}_0 W(c) + \lambda[V_1(C_1^*) - C_1^* - K + c] = 0$$

at the point $c = C_W^*$ and using the definition of $K^*$ we obtain that

$$W(C_W^*) - C_W^* - (V_1(C_1^*) - C_1^* - K) = \frac{\rho}{\lambda + \rho}(K - K^*) + \frac{\rho - r}{\lambda + \rho}(C_0^* - C_W^*)$$

and the desired equivalence now follows from Lemma 23. ■

**Lemma 25** (a) *If $K \geq K^*$ then $W(c) \geq V_1(c - K)$ for all $c \geq K$.*

(b) *If $K < K^*$ then either $V_1(c - K) \geq W(c)$ for all $c \geq K$, or there exists a unique crossing point $C_1^* \leq \tilde{C} \leq C_1^* + K$ such that $V_1(c - K) < W(c)$ if and only if $c < \tilde{C}$.*

**Proof**. We only prove part (b) as both claims follow from similar arguments. Since $W$ is concave by Lemma 19, we have

$$W(c) \leq W(C_W^*) + c - C_W^*$$

and it now follows from Lemma 24 that

$$k(c) \equiv W(c) - V_1(c - K) \leq W(C_W^*) - C_W^* - (V_1(C_1^*) - C_1^* - K) \leq 0.$$

for all $c \geq C_1^* + K$. In order to complete the proof of the first part we distinguish three cases depending on the location of the threshold $C_W^*$.

Case 1: $C_W^* \leq K$. In this case the function $W$ is linear for $c \geq K$. Since $V_1$ is concave, the functions $V_1(c - K)$ and $W(c)$ can have at most 2 crossing points. But, since $V_1(c - K) < W(c)$ for large $c$ as shown above, there can be at most one crossing point.

Case 2: $C_W^* \geq C_1^* + K$. Suppose towards a contradiction that the function $k$ has more than one zero and denote by $z_0 \leq z_1$ its two largest zeros in the interval $[K, C_1^* + K]$. Then, $k(c) > 0$

for $c \in (z_0, z_1)$ due to the above inequality and it follows that the function $k$ has a positive local maximum in the open interval $(z_0, z_1)$. Since $C_W^* \geq C_1^* + K$, it follows from Lemmas 14 and 19 that the function $k$ solves

$$\mathcal{L}_0 k(c) - \lambda k(c) + (-\mu_1 + \mu_0 + rK)V_1'(c - K) = 0 \tag{41}$$

in the interval $[0, C_1^* + K]$ and the required contradiction now follows from Lemma 11 and the fact that $\mu_1 - \mu_0 - rK > 0$ whenever $K \leq K^*$ as a result of Lemma 18.

CASE 3: $C_W^* \in [K, C_1^* + K]$. If $z_1 \leq C_W^*$ then the argument of Case 2 above still applies so assume that the function $k$ does have zeros in the interval $[C_W^*, C_1^* + K]$. Since $V_1(c - K)$ is concave in that interval and $k(C_1^* + K) \leq 0$ we know that the function $k$ can have at most one zero there. Denote the location of this zero by $\bar{z}$ so that $k(c) > 0$ for $c \in [C_W^*, \bar{z})$ and $k(c) \leq 0$ for $c \geq \bar{z}$. Since the function $k$ solves (41) on the interval $[0, C_W^*]$ and satisfies $k(C_W^*) > 0$, $k'(C_W^*) = 1 - V_1'(C_W^* - K) < 0$ it follows from Lemma 12 that $k(c) > 0$ for all $c \leq C_W^*$ and the proof is complete. ■

## D. Proof of Theorem 4

We start this appendix with a standard verification result for the HJB equation associated with the firm's problem:

**Lemma 26** *If $\phi$ is continuous, piecewise twice continuously differentiable and such that*

$$\max\{\mathcal{L}_0\phi(c) + \mathcal{F}\phi(c); 1 - \phi'(c); V_1(c - K) - \phi(c), \ell_0(c) - \phi(c)\} \leq 0\,,$$

*and at each point $c$ at which $\phi'(c)$ jumps, we have $\phi'(c_-) \geq \phi'(c_+)$. Then, $\hat{V}(c) \leq \phi(c)$ for all $c \geq 0$.*

**Proof**. Fix an arbitrary strategy $\pi \in \Pi$, denote by $C_t$ the corresponding cash buffer process and consider the process

$$Y_t = e^{-\rho t}\phi(C_t) + \int_0^t e^{-\rho s}(dD_s - f_{s-}dN_s).$$

Using arguments similar to those of the proof of Lemma 6 it can be shown that $Y_t$ is a local supermartingale[7] and since

$$Z_t = Y_{t\wedge\tau_0} \geq -\int_0^{\tau_0} e^{-\rho s}(dD_s + f_{s-}dN_s)$$

[7]The cases when the derivative jumps are covered by the Ito-Tanaka formula. See Section 3.6 of Karatzas and Shreve (1991) for more details.

where the right hand side is integrable by definition of the set $\Pi$ we conclude that $Z_t$ is a supermartingale. In particular,

$$\begin{aligned}
\phi(c) &= \phi(C_0) - \Delta\phi(C_0) = Z_0 - \Delta\phi(C_0) \geq E_c[Z_\tau] - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau\wedge\tau_0}\phi(C_{\tau\wedge\tau_0}) + \int_{0+}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta\phi(C_0) \\
&= E_c\left[e^{-\rho\tau\wedge\tau_0}\phi(C_{\tau\wedge\tau_0}) + \int_{0}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\
&\geq E_c\left[1_{\{\tau_0\leq\tau\}}e^{-\rho\tau_0}\ell_i(0) + 1_{\{\tau_0>\tau\}}e^{-\rho\tau}V_1(C_\tau)\right] \\
&+ E_c\left[\int_{0}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right] - \Delta D_0 - \Delta\phi(C_0) \\
&\geq E_c\left[1_{\{\tau_0\leq\tau\}}e^{-\rho\tau_0}\ell_i(0) + 1_{\{\tau_0>\tau\}}e^{-\rho\tau}V_1(C_\tau) + \int_{0}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right]
\end{aligned} \tag{42}$$

where the first inequality follows from the optional sampling theorem for supermartingales, the second inequality follows from the assumption of the statement and the last one follows from

$$\Delta D_0 + \Delta\phi(C_0) = \Delta D_0 + \phi(C_{0-} - \Delta D_0) - \phi(C_{0-}) = \int_{C_{0-}-\Delta D_0}^{C_{0-}} (1 - \phi'(c))dc \leq 0\,.$$

Taking the supremum over $\pi \in \Pi$ on both sides of (42) then gives

$$\phi(c) \geq \sup_{\pi\in\Pi} E_c\left[1_{\{\tau_0\leq\tau\}}e^{-\rho\tau_0}\ell_i(0) + 1_{\{\tau_0>\tau\}}e^{-\rho\tau}V_1(C_\tau) + \int_{0}^{\tau\wedge\tau_0} e^{-\rho s}(dD_s - f_{s-}dN_s)\right]$$

and the result now follows from Lemma 16. ■

**Lemma 27** *If $U'(0) \geq W'(0)$ then $U$ satisfies the conditions of Lemma 26 and $C_U^* \leq C_1^* + K$.*

**Proof**. First of all, if $\psi'(K;K) \geq V_1'(0)$, then we have $C_U^* = K$ from Lemma 21 and we only need to show that $U'(c) = \psi'(c;K) \geq 1$ for $c \leq K$. To this end, let $\bar{W}$ be the unique solution to

$$\mathcal{L}_0\bar{W}(c) - \lambda\bar{W}(c) + \lambda(V_1(C_1^*) - C_1^* + c) = 0, \qquad c \geq 0,$$

which coincides with the function $W$ on the interval $[0, C_W^*]$. Since $\bar{W}$ satisfies $\bar{W}'(C_W^*) = 1$ and $\bar{W}''(C_W^*) = 0$, it follows from Lemma 13 that $\bar{W}'(c) \geq \bar{W}'(C_W^*) = 1$ for all $c \geq 0$. Then, the difference $m(c) = \psi(c;K) - \bar{W}(c)$ satisfies

$$\mathcal{L}_0 m(c) - \lambda m(c) = 0, \qquad c \in [0, K]\,. \tag{43}$$

Furthermore, $m(0) = 0$ and $m'(0) \geq 0$ since, by assumption $U'(0) = \psi'(0;K) \geq W'(0) = \bar{W}'(0)$. Lemma 12 implies that $m'(c) \geq 0$ that is $\psi'(c;K) \geq \bar{W}'(c) \geq 1$, which is what had to be proved.

Now assume that $\psi'(K;K) < V_1'(0)$ so that $C_U^* > K$. In order to show that $U'(c) \geq 1$, consider the function $\phi(c) = U(c) - \bar{W}(c)$. By Lemmas 19, 21 we have that the function $\phi$ solves (43) and, since $\phi(0) = 0$ and $\phi'(0) = U'(0) - W'(0) > 0$ by assumption, it follows from Lemma 11 that $\phi'(c) \geq 0$ and consequently

$$U'(c) \geq \bar{W}'(c) \geq 1, \qquad c \leq C_U^*. \tag{44}$$

Using (44) in conjunction with the definition of the liquidation value, it is immediate to show that

$$U(c) = U(0) + \int_0^c U'(x)dx = \ell_0(0) + \int_0^c U'(x)dx \geq \ell_0(0) + c = \ell_0(c).$$

The inequality $U(c) \geq V_1(c-K)$ is contained in the proof of Lemma 21 and the proof that $U$ satisfies the conditions of Lemma 26 will be complete once we show that $\mathcal{L}_0 U(c) + \mathcal{F}U(c) \leq 0$. A direct calculation using the fact that, as shown below, $C_U^* \leq C_1^* + K$ together with the definition and concavity of the functions $U$ and $V_1$ shows that

$$\mathcal{L}_0 U(c) + \mathcal{F}U(c) = \begin{cases} 0, & c \leq C_U^*, \\ (rK - \mu_1 + \mu_0)V_1'(c-K), & C_U^* \leq c \leq C_1^* + K, \\ (r-\rho)(c - (C_1^* + K)) + \mu_0 - \mu_1 + rK, & c \geq C_1^* + K. \end{cases}.$$

and the desired result now follows from the increase of $V_1$ and the fact that $\mu_0 - \mu_1 + rK < 0$ for all $K \leq K^*$ by Lemma 18.

In order to show that $C_U^* \leq C_1^* + K$ assume towards a contradiction that $C_U^* > C_1^* + K$. In this case we have that $U'(C_U^*) = 1$ and, since $U(c) > V_1(c-K)$, we get that $U(c)$ is convex in a small neighborhood of $C_U^*$. This implies that $U'(c) < U'(C_U^*) = 1$ for $c$ in this small neighborhood, which is impossible die to the first part of the proof. ■

Having dealt with the case where the firm uses exclusively the strategy (U), we now turn to the case in which the firm mixes the strategies (U) and (W). To state the result, recall that the function $v_1(c-K)$ is defined by (37).

**Lemma 28** *Assume that $U'(0) < W'(0)$. Then the unique solution to the free boundary problem defined by (21), (16), (22), (23), (24) is given by*

$$V(c) = \begin{cases} W(c), & c \leq C_L^*, \\ S(c), & C_L^* \leq c \leq C_H^*, \\ V_1(c-K), & c \geq C_L^*, \end{cases} \tag{45}$$

*where the function $S$ is defined by (25) and the constants $C_L^* \in [C_W^*, \tilde{C})$, $C_H^* \in [C_U^*, C_1^* + K]$ are* *the unique solutions to the value matching and smooth pasting condition*

$$\begin{aligned} S(C_L^*) &= \xi_G(C_H^*)F_0(C_L^*) - \xi_F(C_H^*)G_0'(C_L^*) + \Phi(C_L^*) \\ &= W(C_L^*), \end{aligned} \tag{46}$$

$$\begin{aligned} S'(C_L^*) &= \xi_G(C_H^*)F_0'(C_L^*) - \xi_F(C_H^*)G_0'(C_L^*) + \Phi'(C_L^*) \\ &= W'(C_L^*) = 1. \end{aligned} \tag{47}$$

*Furthermore,* $\max\{W(c)\,,\,V_1(c-K)\} \;\le\; V(c) \;\le\; \hat{V}(c)$ *for all* $c \ge 0$*.*

**Proof**. Using arguments similar to those of the proof of Lemma 7, it can be shown that the unique solution to (21) such that (22) and (24) holds is given by (45) and the first part of the proof will be complete once we show that (46) and (47) admit unique solutions.

By Lemma 22, finding a solution to (21), (16), (22), (23), (24) is equivalent to finding a linear function that is tangent to the graph of the functions $\hat{w}$ and $\hat{v}_1$ defined by

$$w(c) = W(c) - \Phi(c) = F_0(c)\hat{w}(Z(c)) = F_0(c)\hat{w}\left(\frac{G_0(c)}{F_0(c)}\right).$$

and (39). A direct calculation using the results of Lemma 19 shows that

$$\mathcal{L}_0 w(c) - \lambda w(c) = (r-\rho)(c - C_W^*)^+$$

and it now follows from Lemma 22 that the function $\hat{w}$ is linear for $y \le Z(C_W^*)$ and concave otherwise. Since $W(c)$ is concave by Lemma 19, we get

$$w'(c) = W'(c) - \frac{\lambda}{\lambda+\rho-r} \ge W'(C_W^*) - \frac{\lambda}{\lambda+\rho-r} = \frac{\rho-r}{\lambda+\rho-r} > 0.$$

Since $F_0$ is nonnegative and decreasing, the ratio $w(c)/F_0(c)$ is positive and strictly increasing for sufficiently large $c$. Therefore, by Lemma 22, $\hat{w}(y)$ is increasing for sufficiently large $y$ and, since $\hat{w}(y)$ is concave, it is globally increasing.

Since

$$U(0) = W(0) = \ell_0,$$

we obtain that both $\hat{w}$ and the function

$$\hat{u}(y) = (U(Z^{-1}(y)) - \Phi(Z^{-1}(y)))/F_0(Z^{-1}(y))$$

are linear on $[Z(0), Z(C_W^*) \wedge Z(C_U^*)]$ and coincide at $Z(0)$. The inequality $U'(0) < W'(0)$ implies that $\hat{u}(y) \le \hat{w}(y)$ for all $y \in [Z(0), Z(C_W^*) \wedge Z(C_U^*)]$. It follows that $C_W^* \le \tilde{C} < C_U^*$ because $\hat{w}$ is a linear function that crosses the graph of the concave function $\hat{v}_1(y)$ at the point $Z(\tilde{C})$ and, by the definition of $\tilde{C}$, we have $\hat{w}(y) < \hat{v}_1(y)$ for all $y > Z(\tilde{C})$.

Since

$$U(c) \geq V_1(c-K), \qquad c \leq C_U^*$$

by Lemma 21, we get that the linear function

$$\bar{w}(y) = \frac{\hat{w}(Z(C_W^*)) - \hat{w}(Z(0))}{Z(C_W^*) - Z(0)} y + \frac{\hat{w}(Z(0))Z(C_W^*) - \hat{w}(Z(C_W^*))Z(0)}{Z(C_W^*) - Z(0)}$$

is tangent to the concave function $\hat{w}(y)$ and lies strictly above the concave function $v_1(y)$ for all $y \geq Z(\tilde{C})$. On the other hand, since

$$\hat{v}_1(Z(\tilde{C})) = \hat{w}(Z(\tilde{C})),$$
$$\hat{v}_1'(Z(\tilde{C})) > \hat{w}'(Z(\tilde{C}))$$

as a result of Lemma 25, we have that the tangent line to $\hat{w}$ at the point $y = Z(\tilde{C})$ lies strictly below $v_1$ for $y > Z(\tilde{C})$. By continuity, this implies that there exists a unique point $y_L^* \in (Z(C_W^*), Z(\tilde{C}))$ such that the tangent line to $\hat{w}$ at $y_L^*$ is also tangent to $\hat{v}_1$ at some $y_H^* > y_L^*$. Setting

$$C_L^* = Z^{-1}(y_L^*) < Z^{-1}(y_H^*) = C_H^*$$

produces the unique solution to (46), (47) and it now only remains to show that $C_U^* < C_H^* < C_1^* + K$. Since $\hat{w}$ is is increasing and concave its tangent line at the point $y_L^*$ crosses the vertical axis above the level $\hat{w}(Z(0))$. However, if $C_U^*$ was larger than $C_H^*$ then this tangent would have to cross the vertical axis below $\hat{u}(Z(0)) = \hat{w}(Z(0))$ thus leading to a contradiction. Furthermore, since $\hat{w}$ and $\hat{v}_1$ are both concave, we get that $\hat{v} \geq \max\{\hat{w}, \hat{v}_1\}$ and therefore $V(c) \geq \max\{W(c)\,,\, V_1(c-K)\}$. The claim $C_H^* > C_1^* + K$ follows from Lemma 29 below.

In order to show that $V(c) \leq \hat{V}(c)$ and thus complete the proof let $\tau_L^*$ (resp. $\tau_H^*$) denote the first time that the firm's cash reserves falls below $C_L^*$ (resp. above $C_H^*$). Using arguments similar to those of the proof of Lemma 27 it can be shown that

$$\begin{aligned} V(c) = E_c\Big[ & 1_{\{\tau_H^* < \tau_N \wedge \tau_L^*\}} e^{-\rho \tau_H^*} V_1(C_H^* - K) + 1_{\{\tau_L^* < \tau_N \wedge \tau_H^*\}} e^{-\rho \tau_L^*} W(C_L^*) \\ & + 1_{\{\tau_N < \tau_L^* \wedge \tau_H^*\}} e^{-\rho \tau_N} (V_1(C_1^*) - C_1^* - K + C_{\tau_N -})\Big]. \end{aligned}$$

On the other hand, using arguments similar to those of the proof of Proposition 1 it can be shown that the function $W$ satisfies

$$W(c) = E_c\left[1_{\{\tau_0 < \tau_N\}} e^{-\rho\tau_0} \ell_0 + 1_{\{\tau_0 \geq \tau_N\}} e^{-\rho \tau_N} V_1(C_1^*) + \int_0^{\tau_0 \wedge \tau_N} e^{-\rho s}(dL_s - f_{s-}^U dN_s)\right]$$

where $L_t = \sup_{s \leq t} (X_t - C_W^*)^+$ with

$$dX_t = (rX_{t-} + \mu_i)dt + \sigma dB_t + (C_W^* - X_{t-})^+ dN_t\,,$$

and $f^U$ is defined in (40). Combining these two equalities and using the law of iterated expectations then gives

$$V(c) = E_c\left[1_{\{\tau_0<\tau\}}e^{-\rho\tau_0}\ell_0 + 1_{\{\tau_0\geq\tau\}}e^{-\rho\tau}V_1(C_\tau) + \int_0^{\tau_0\wedge\tau} e^{-\rho s}(dD_s^V - f_{s-}^U dN_s)\right]$$

where $\tau = \tau_N \wedge \tau_H^*$ and the cumulative dividend process is defined by

$$D_t^V = \int_0^t 1_{\{C_{s-}\leq C_L^*\}}dL_s.$$

As is easily seen the strategy $(\tau, D^V, f^U)$ is admissible and the desired result now follows from Lemma 16 by taking the supremum over the set of admissible strategies on both sides. ■

**Lemma 29** *If $U'(0) < W'(0)$, then $V$ satisfies the conditions of Lemma 26 and $C_H^* < C_1^* + K$.*

**Proof**. In order to show that $V' \geq 1$ we start by observing that this inequality holds in $[0, C_L^*] \cup [C_H^*, \infty)$ due to the definition of the function $V$ and Lemmas 14, 19, 21. Now, since $C_H^* \geq C_L^* \geq C_W^*$, we have $V'(C_L^*) = W(C_L^*) = 1$ and

$$V(c) \geq W(c) = W(C_W^*) + c - C_W^*, \qquad C_L^* \leq c \leq C_H^*.$$

This immediately implies that $V''(C_L^*) \geq 0$ and since $J(c) = V'(c)$ is a solution to

$$\frac{\sigma^2}{2}J''(c) \;+\; (rc+\mu_0)J'(c) - (\lambda+\rho-r)J(c) + \lambda \;=\; 0,$$

it follows from Lemma 11 that $J' = V''$ can have at most one zero in the interval $\mathcal{I} = [C_L^*, C_H^*]$. If no such zero exists then $V'' \geq 0$ in $\mathcal{I}$ and consequently $V'(c) \geq V'(C_L^*) = 1$ for all $c \in \mathcal{I}$. If on the contrary $V''$ has one zero located at some $c^* \in \mathcal{I}$ then we have that $V'$ reaches a global maximum over $\mathcal{I}$ at the point $c^*$ and since $V'(C_H^*) = V_1'(C_H^* - K) \geq 1$ due to the concavity of $V_1$, we conclude that the inequality $V'(c) \geq 1$ holds for all $c \in \mathcal{I}$.

Let us now show that $C_H^* \leq C_1^* + K$. Indeed, if this is not the case, we have $V'(C_H^*) = 1$. Since $V(c) > V_1(c-K)$, we have that $V(c)$ is convex in a small neighborhood of $C_H^*$ and therefore $V'(c) < V'(C_H^*) = 1$ for $c$ in this small neighborhood, which is impossible by the previous paragraph.

Using the fact that this inequality holds for all $c$ in conjunction with the definition of the liquidation value it is immediate to show that

$$V(c) = V(0) + \int_0^c V'(x)dx = \ell_0(0) + \int_0^c V'(x)dx \geq \ell_0(0) + c = \ell_0(c).$$

The fact that $V(c) \geq \max\{W(c)\,,\, V_1(c-K)\}$ is contained in Lemma 28.

Since $C_W^* \leq C_L^* \leq C_H^* \leq C_1^* + K$ , a direct calculation using this property together with the definition and concavity of the functions $W$ and $V_1$ shows that

$$\mathcal{L}_0 V(c) + \mathcal{F}V(c) = \begin{cases} 0, & c \leq C_W^*, \\ (r-\rho)(C - C_W^*), & C_W^* \leq c \leq C_L^*, \\ 0, & C_L^* \leq c \leq C_H^*, \\ AV_1'(c-K) + (r-\rho)(c - C_1^* - K)^+, & c \geq C_H^*, \end{cases}$$

where we have set $A = \mu_0 - \mu_1 + rK$. By Lemma 17 we know that $A \geq 0$ whenever $K \leq K^*$ and it thus follows from the increase of the function $V_1$ that

$$\mathcal{L}_0 V(c) + \mathcal{F}V(c) \leq 0, \qquad c \geq 0.$$

The proof is complete. ■

**Lemma 30** *There exists a unique $K^{**} \in (0, K^*)$ such that $U'(0) > W'(0)$ if and only if $K < K^{**}$.*

**Proof**. We know from the proof of Lemma 31 that $U'(0) = \psi'(0; K) > W'(0)$ for sufficiently small $K$. Similarly, for $K = K^*$ we know that $V_1(c - K^*)$ touches $W(c)$ from below at $C_1^* + K$ so that $C_L^* = C_H^* = C_1^* + K$ and $U'(0) < W'(0)$. Thus, it suffices to show that there exists a single point $K^{**} \in [0, K^*]$ such that

$$U'(0; K^{**}) = W'(0; K^{**}) \,.$$

Assume towards a contradiction that this is not the case so that there exist two points $K_1 < K_2$ such that $U'(0; K_1) = W'(0; K_1)$ and $U'(0; K_2) = W'(0; K_2)$. Let the function $\bar{W}_i$ denote the unique solution to

$$\mathcal{L}_0 \bar{W}_i(c) - \lambda \bar{W}_i(c) + \lambda(V_1(C_1^*) - C_1^* - K_i + c) = 0, \qquad c \geq 0,$$

which coincides with $W(\cdot; K_i)$ on the interval $[0, C_W^*(K_i)]$ and recall from the proof of Lemma 27 that this function is concave for $c \leq c_i^* = C_W^*(K_i)$ and convex for $c \geq c_i^*$ so that $\bar{W}_i'(c) > \bar{W}'(c_i^*) = 1$ for all $c \neq c_i^*$.

Since $U(c; K_i)$ and $\bar{W}_i(c)$ have the same value at $c = 0$, $U'(0; K_i) = W'(0; K_i)$ implies that $U(c; K) = \bar{W}_i(c)$ for $c \leq C_U^*(K_i)$ . Consider the function $m(c) = \bar{W}_i(c) - V_1(c - K_i)$. Then, $m(c)$ satisfies (41). If $C_U^*(K_i) > K_i$, then $m(C_U^*(K_i)) = m'(C_U^*(K_i)) = 0$ and it follows from the proof of Lemma 27 that $m(c) \geq 0$ for all $c \geq K_i$. If $C_U^*(K_i) = K_i$ then $\bar{W}_i'(C_U^*(K_i)) \geq V_1'(0)$ implies that $m(C_U^*(K_i)) = 0$ , $m'(C_U^*(K_i)) \geq 0$ and therefore $m(c) \geq 0$ for $c \geq K_i$ by Lemma 12

The function

$$k(c) = \bar{W}_2'(c) - \bar{W}_1'(c)$$

is a solution to

$$(rc+\mu_0)k'(c)+\frac{\sigma^2}{2}k''(c)-(\rho+\lambda-r)k(c)=0, \qquad c\geq 0,$$

and satisfies $k(c_2^*)>0$, $k(c_1^*)<0$. Since $k(c)$ cannot have local negative minima by Lemma 11, it follows that there exists a unique point $c_*\in(c_2^*,c_1^*)$ such that $k(c_*)=0$, $k'(c_*)>0$ and $k(c)>0$ for all $c>c_*$ and $k(c)<0$ for $c<c_*$. That is, $\bar W_2-\bar W_1$ attains its global minimum at $c_*$ and $(\bar W_2-\bar W_1)''(c_*)>0$. Evaluating the equation

$$\frac{1}{2}\sigma^2\,(\bar W_2-\bar W_1)''(c)+(rc+\mu_0)(\bar W_2-\bar W_1)'(c)-(\rho+\lambda)(\bar W_2-\bar W_1)(c)+\lambda(K_1-K_2)\;=\;0$$

at $c_*$, we get

$$(\bar W_2-\bar W_1)(c_*)\;>\;\frac{\lambda}{\rho+\lambda}(K_1-K_2)\,,$$

and therefore

$$(\bar W_1-\bar W_2)(c)\;<\;\frac{\lambda}{\rho+\lambda}(K_2-K_1)$$

for all $c$. However, since, by the above, $\bar W_1(c)\;\geq\;V_1(c-K_1)$ for $c\geq K_1$ and $V_1'\geq 1$, we get

$$\begin{aligned}\frac{\lambda}{\rho+\lambda}(K_2-K_1)\;&\geq\;W_1(C_H^*)-W_2(C_H^*)=W_1(C_H^*)-V_1(C_H^*-K_2)\\&\geq V_1(C_H^*-K_1)-V_1(C_H^*-K_2)\geq K_2-K_1,\end{aligned}$$

which is a contradiction. ■

**Lemma 31** *There exists a unique* $\underline{K}\in(0,K^{**})$ *such that, for* $K\in(0,K^{**})$*, we have* $\psi'(K,K)<V_1'(0)$ *if and only if* $K>\underline{K}$.

**Proof**. First of all, we show that $\lim_{K\downarrow 0}\psi'(K;K)=+\infty$. Indeed,

$$G_0(K)F_0(0)-F_0(K)G_0(0)\;\approx\;K\,(G_0'(0)F_0(0)-F_0'(0)G_0(0))=:\alpha K$$

with $\alpha>0$. Therefore,

$$\psi'(K;K)\;\approx\;(\alpha K)^{-1}(G_0(0)(\ell_0-\ell_1)F_0'(0)-F_0(0)(\ell_0-\ell_1)G_0'(0))=K^{-1}(\ell_1-\ell_0)$$

and the required assertion follows from the fact that $\ell_1\geq\ell_0$.

Since $\psi'(K;K)$ is continuous in $K$, it remains to show that the equation $\psi'(K;K)$ can have at most one solution $\underline{K}$. Suppose the contrary and fix $K_1<K_2$ such that $\psi'(K_1;K_1)=\psi'(K_2;K_2)=V_1'(0)$. Let $\psi_i(c)=\psi(c;K_i)$. Then,

$$\mathcal{L}_0\psi_i(c)-\lambda\psi_i(c)+\lambda(V_1(C_1^*)-C_1^*-K_i+c)\;=\;0\;,\;\;c\geq 0.$$

By Lemma 27, $\psi'(c;K) \geq 1$ since $K \leq K^{**}$. Now, consider the functions $\tilde{\psi}_i(y) = \psi_i(y+K_i)$. Then,

$$0.5\sigma^2\tilde{\psi}_i''(y) + (ry + rK_i + \mu_0)\tilde{\psi}_i'(y) - (\rho+\lambda)\tilde{\psi}_i(y) + \lambda(V_1(C_1^*) - C_1^* + y) \; = \; 0\,.$$

Let $m(c) = \tilde{\psi}_1(y) - \tilde{\psi}_2(y)$. Then, $m(0) = \psi_1(K_1) - \psi_2(K_2) = 0,\; m'(0) = \psi_1'(K_1) - \psi_2'(K_2) = 0$. Furthermore,

$$0.5\sigma^2 m''(y) + (ry + rK_1 + \mu_0)m'(y) - (\rho+\lambda)m(y) + r(K_1 - K_2)\tilde{\psi}_2'(y) = 0\,.$$

Since $\tilde{\psi}_2'(y) > 0$, Lemma 12 implies that $m(c)$ is positive and monotone decreasing and, consequently, $\tilde{\psi}_1(y) > \tilde{\psi}_2(y)$ for $y < 0$. Therefore, since $\tilde{\psi}_2$ is monotone decreasing, we get

$$\ell_0 = \tilde{\psi}_1(-K_1) > \tilde{\psi}_2(-K_1) > \tilde{\psi}_2(-K_2) = \ell_0,$$

which is a contradiction. ■

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Figure 1: Cash holdings for a firm with no growth option.

Figure 1 plots the value-maximizing cash buffer when the firm has no access to external funds (blue line) and when the firm has access to external funds (dashed red line) as a function of the arrival rate of investors $\lambda$, the reinvestment rate $r$, the recovery rate on assets $\varphi$, and cash flow volatility $\sigma$. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, and $\mu = .1$.

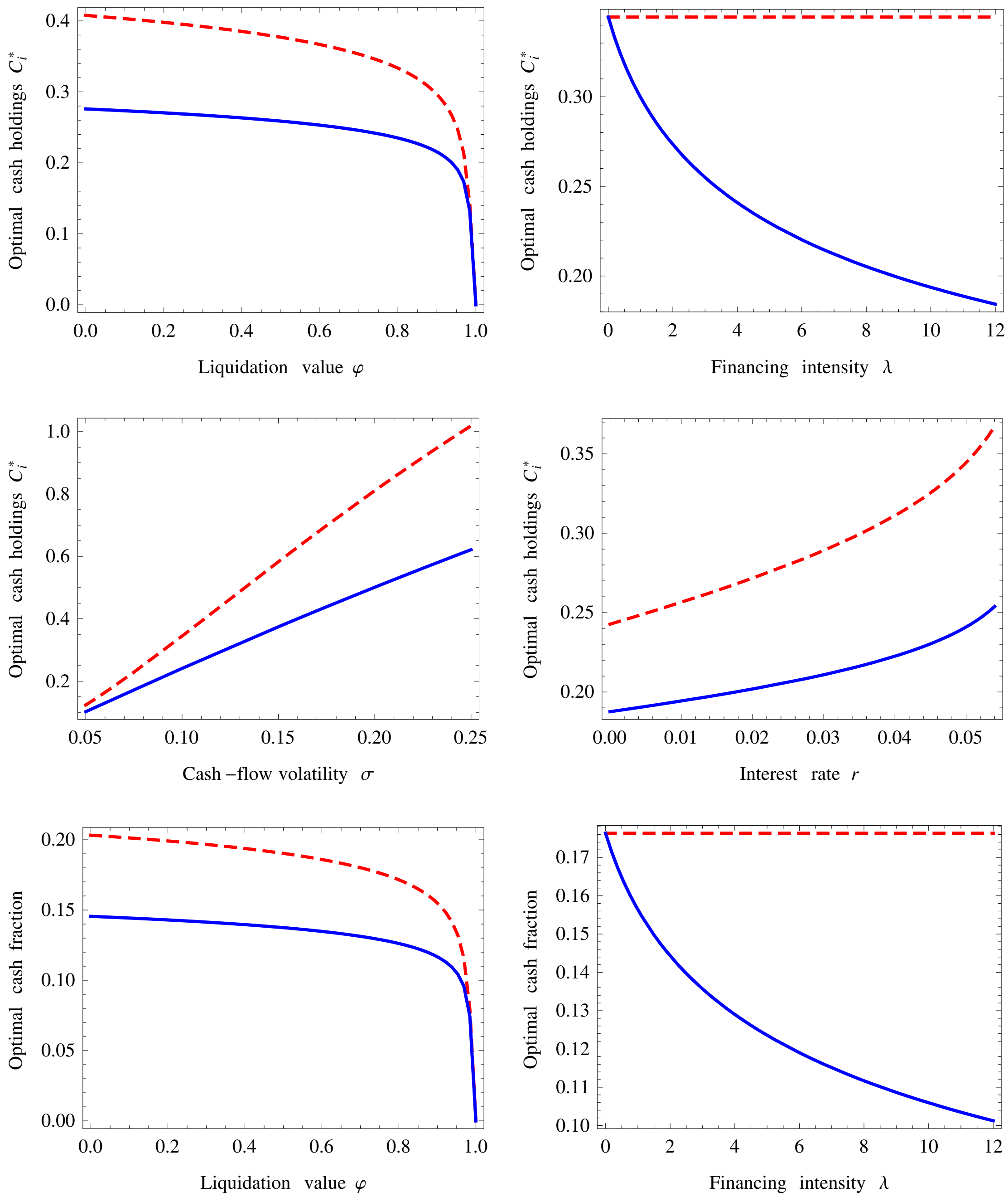


Figure 2: Firm value before investment.

Figure 2 plots the value of the firm before investment. When $C_L^* = 0$, the optimal policy is to finance investment with internal funds at the point $C_U^*$ or with external funds (left panel). When $C_L^* > 0$, the optimal policy for the firm is to finance investment with external funds when cash holdings are below $C_L^*$ and otherwise to finance investment with internal funds at the point $C_H^*$ or with external funds (right panel).

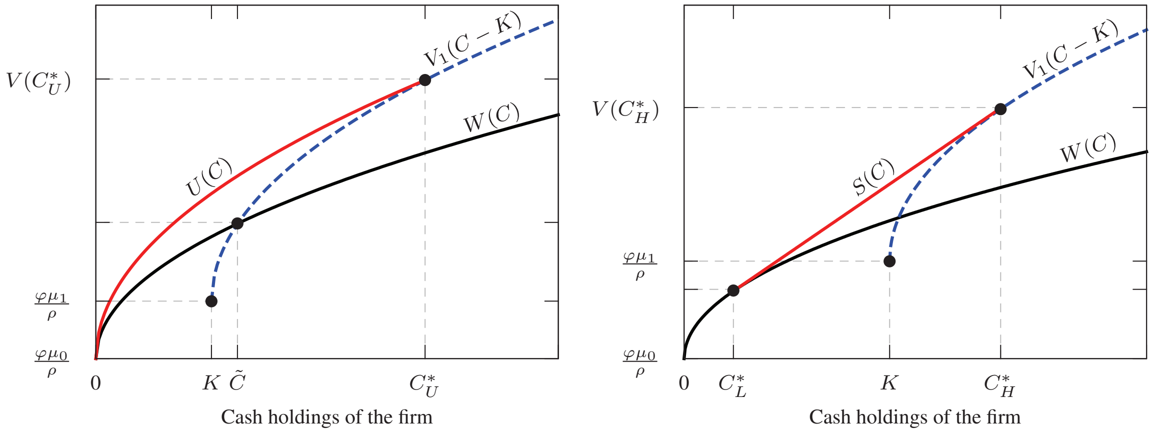
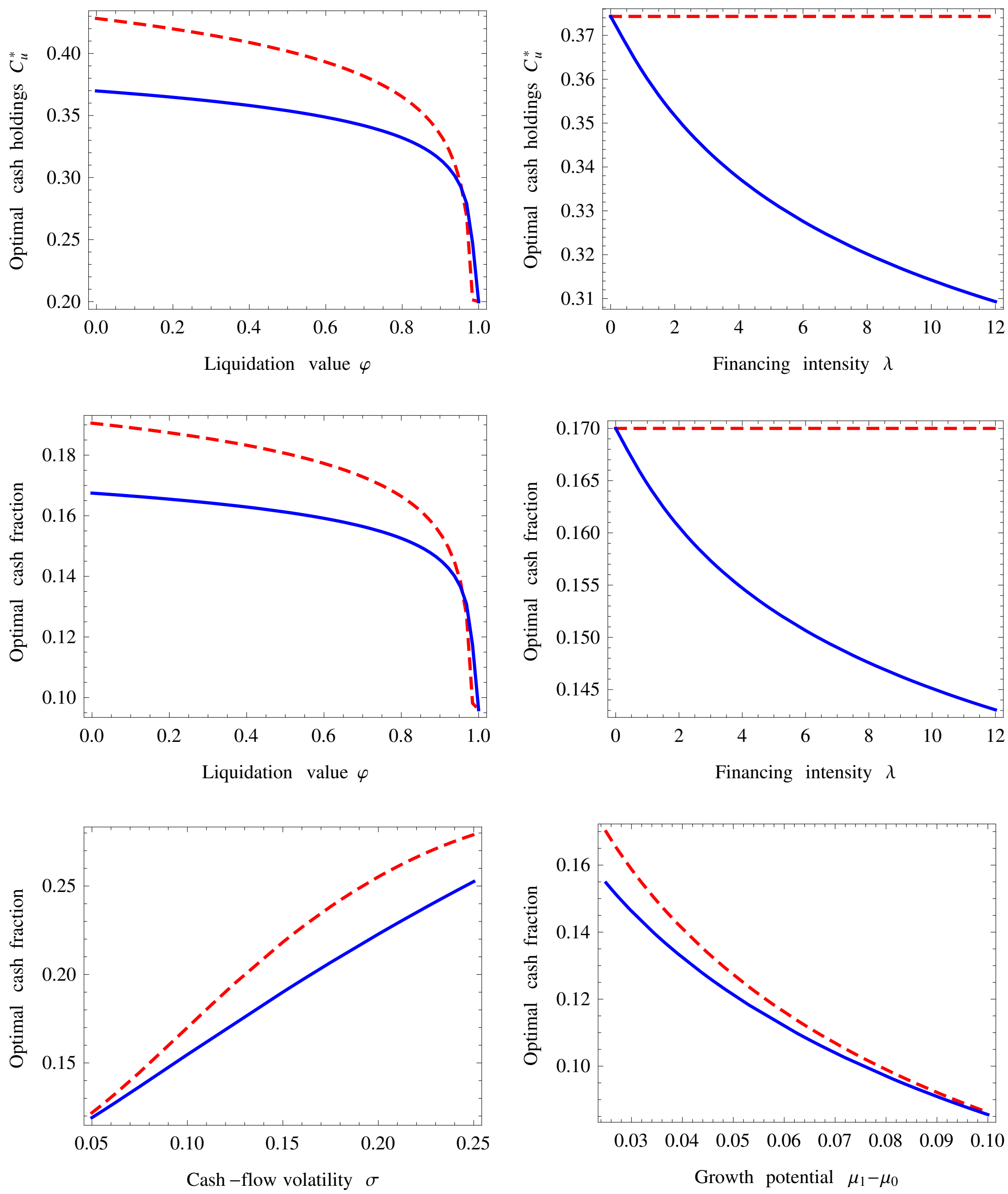


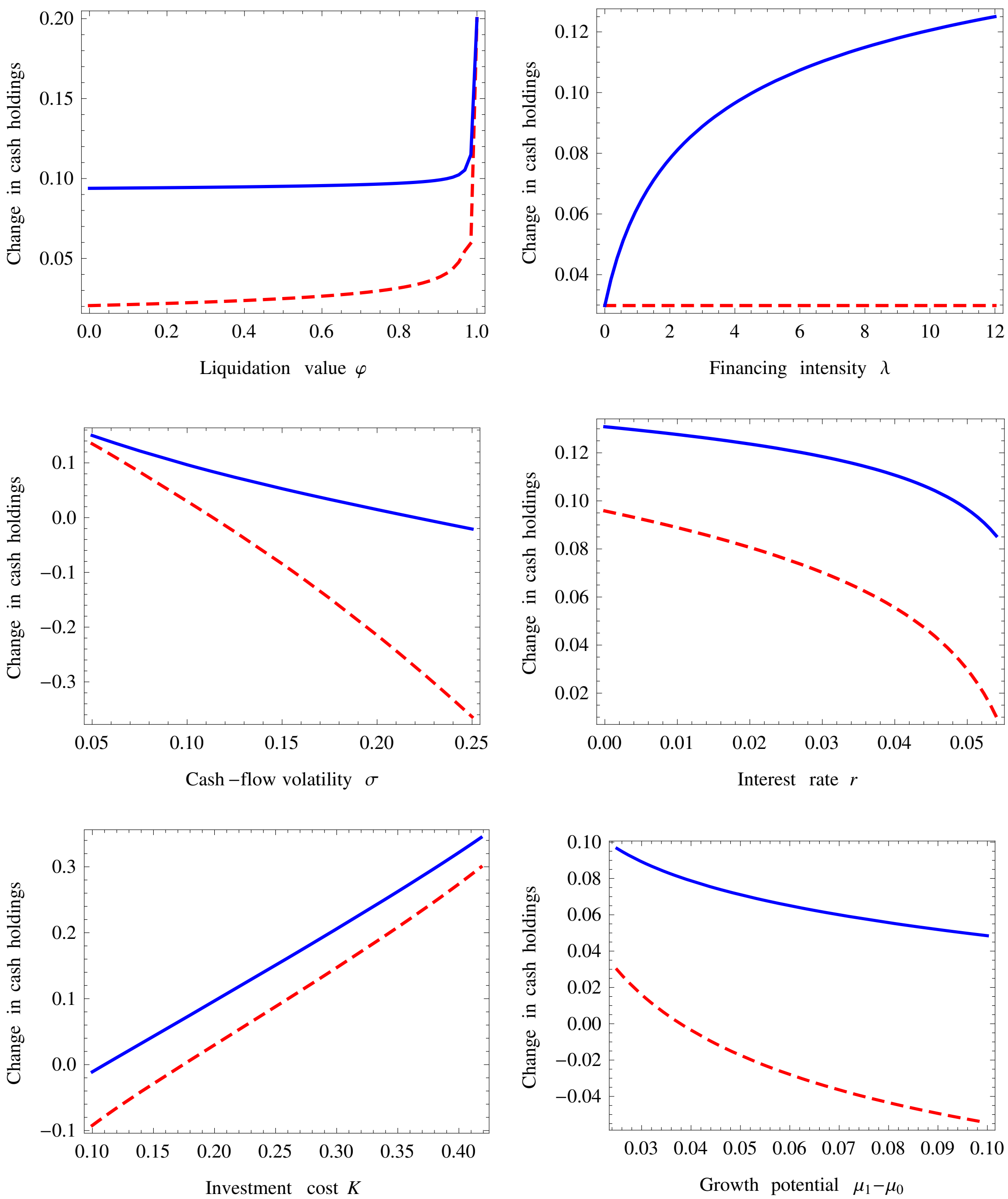
Figure 3: Cash holdings for a firm with a growth option.

Figure 3 plots the optimal cash buffer when the firm has no access to external funds (blue line) and when the firm has access to external funds (dashed red line) as a function of the arrival rate of investors $\lambda$, the recovery rate on assets $\varphi$, cash flow volatility $\sigma$, and the growth potential of the firm $\mu_1 - \mu_0$. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, and $\mu_0 = .1$.

Panel A: Optimal Cash buffer Before Investment.



Panel B: Change in the Optimal Cash buffer.



## Figure 4: Probability of Investment.

Figure 4 plots the probability of investment using internal funds (red dashed line) and investment using external funds (solid blue line) as a function of the arrival rate of investors $\lambda$, the reinvestment rate $r$, the recovery rate on assets $\varphi$, cash flow volatility $\sigma$, the cost of investment $K$, and the growth potential of the firm $\mu_1 - \mu_0$. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, and $\mu_0 = .1$.

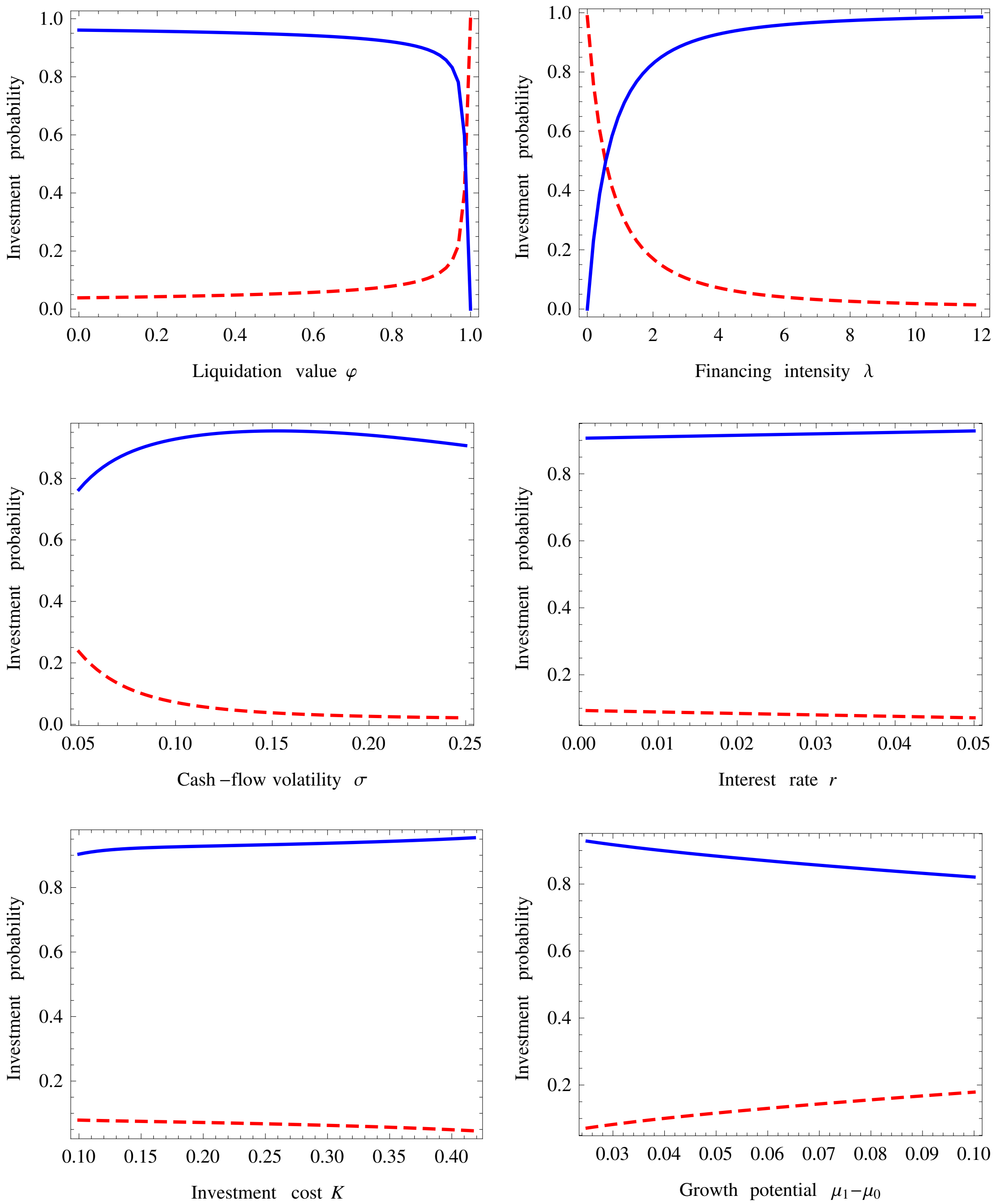


Figure 5: Capital Supply and Investment.

Figure 5 plots the probability of investment using internal funds, the probability of investment using external funds, and the probability of liquidation over a 1-year (red dotted line) and 3-year (solid blue line) horizon as a function of the arrival rate of investors $\lambda$. The base parametrization is $\rho = .06$, $r = .05$, $\lambda = 4$, $\varphi = .75$, $\sigma = .1$, and $\mu_0 = .1$.