Disagreement about Inflation and the Yield Curve[*]

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Abstract

We study how differences in beliefs about expected inflation impact real and nominal yield curves in a frictionless economy. Inflation disagreement induces a spillover effect to the real side of the economy with a strong impact on the real yield curve. When investors have a coefficient of relative risk aversion greater than one, real yields across all maturities rise as disagreement increases. Real yield volatilities also rise with disagreement. Using the feature that nominal bond prices can be computed from weighted-averages of quadratic Gaussian yield curves, we explore three properties of the model numerically. First, both real and nominal yield curves are strongly impacted by inflation disagreement relative to a full information economy. Second, increased inflation disagreement drives nominal yields and nominal yield volatilities higher at all maturities. Third, expected inflation beliefs impact real yields. Empirical support for our predictions on yield levels and yield volatilities is provided.

Keywords: Disagreement about expected inflation, feedback effect, real and nominal yields.

JEL Classification: D51, E43, E52, G12.

1   Introduction

Several sophisticated reduced-form term structure models exist that are successful in explaining empirical features of U.S. Treasury bonds.[1] However, the economic mechanisms driving these empirical regularities are not well understood.[2] One logical candidate is how market participants view inflation. Indeed, since Friedman's and Phelps' works in the 1960s, inflation expectations have defined the core of monetary policy work and so naturally should impact bond prices. Mankiw, Reis, and Wolfers (2004) later argued that *disagreement about inflation expectations* "may be a key to macroeconomic dynamics." This is the departure point for our work where we study the role that disagreement about expected inflation plays in determining properties of real and nominal yield curves in a frictionless economy. Our contribution is to show both theoretically and empirically how disagreement impacts yield levels and volatilities for both real and nominal yield curves.

A key feature of our model is that disagreement about expected inflation impacts the equilibrium real pricing kernel. This spillover effect from the nominal to the real side of the economy is generated by investors engaging in speculative trade in nominal bonds due to their differing opinions on inflation. Even if inflation is uncorrelated with economic fundamentals, the real pricing kernel is impacted by inflation disagreement — a feature like a sunspot equilibrium in Cass and Shell (1983) or Basak (2000). This mechanism of nominal quantities impacting real quantities is distinct from New-Keynesian monetary models such as Clarida, Galí, and Gertler (1999) where spillover effects occur through sticky prices.

We find that disagreement about expected inflation significantly impacts the level of the real yield curve. The direction of the shift on the real yield curve level is driven by the relative strength of income and substitution effects. When investors have a coefficient of relative risk aversion greater than one, real average yields across all maturities rise as disagreement increases. When the

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[1]Examples of reduced-form no arbitrage models with latent state variables include Dai and Singleton (2000, 2002, 2003), Duffee (2002), and Ahn, Dittmar, and Gallant (2002). See Singleton (2006) and Piazzesi (2010) for an overview of this literature.

[2]Building from the reduced-form no arbitrage models, an intermediate approach has been to introduce macroeconomic variables into these no-arbitrage settings as in Ang and Piazzesi (2003), Ang, Dong, and Piazzesi (2007b), and Ang, Bekaert, and Wei (2008). Other works have explored full structural models. Wachter (2006), Piazzesi and Schneider (2007), and Bansal and Shaliastovich (2012) study structural term structure models with exogenous inflation. Basak and Yan (2010) incorporate money illusion in a setting with exogenous inflation. Buraschi and Jiltsov (2005), Gallmeyer, Hollifield, and Zin (2005), Gallmeyer, Hollifield, Palomino, and Zin (2007), Geanakoplos et al. (2009), Bekaert, Cho, and Moreno (2010), and Palomino (2012) study structural models that endogenize inflation. The recent surveys by Gürkaynak and Wright (2012) and Rudebusch (2010) summarize the implications of some of this structural work.

coefficient of relative risk aversion is less than one, real yield levels fall as disagreement increases. Yield volatilities always increase with disagreement.

What is the intuition for the relation between yields and disagreement? Investors with different beliefs about inflation trade nominal bonds to increase their consumption share in the future. Focusing on the case when the relative risk aversion is above one given it better explains term structure properties, the income effect dominates. For this case, investors wish to increase their consumption today. Since consumption is exogenous, it is not possible for all investors to increase consumption. Market clearing implies that the short rate increases to counter balance consumption demands. Importantly, we prove that this increase in the short rate propagates through the entire yield curve.

To derive implications for nominal yield curves as well as to quantify the impact on real yield curves, we numerically study the impact of disagreement about expected inflation in a setting where bond prices can be computed from weighted-averages of quadratic Gaussian term structure models. To help match the level and the slope of yields as well as yield volatilities, we employ external habit-formation preferences as in Abel (1990, 1999) and Chan and Kogan (2002) because otherwise inflation disagreement cannot match these curves with plausible parameters. We study three aspects of how inflation disagreement impacts real and nominal yield curve properties.

First, we ask how disagreement impacts means and volatilities of yields relative to learning and full information economies. When the steady state difference in expected inflation across investors is only 30 basis points, real yield curves are shifted up by approximately 150 basis points while yield volatilities are increased by approximately 22% relative to a full information economy. The impact on nominal yield curves is even stronger as yield curves increase by approximately 200 basis points and yield volatilities increase by approximately 28% relative to the fully information economy. These effects are driven by the disagreement economy and not by single investor learning economies as those economies exhibit much smaller impacts relative to the full information economy.

Second, we study how real and nominal yield curves are impacted by increased disagreement and the consumption sharing rule. We numerically show that when disagreement about expected inflation increases, nominal yields rise, the nominal yield curve flattens, and nominal yield volatilities increase. These impacts can be large. For a reasonable increase in disagreement, nominal yields at short maturities can rise by as much as 150 basis points. Similar behavior occurs for the

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real yield curve. We also demonstrate that these effects are amplified when investors with differing expected inflation views have similar consumption shares.

Third, disagreement about expected inflation provides a channel for the level of inflation expectations to impact real yield curves. When investors disagree about the persistence of expected inflation, then current expected inflation predicts future disagreement about inflation and thus future consumption and wealth shares. Hence, real yield curves are strongly impacted by inflation expectations as maturities above the short rate exhibit a strong U-shaped pattern as a function of expected inflation.

We use the Michigan Survey of Consumers to compute a measure of differences in beliefs. We find support for increased inflation belief dispersion leading to higher real yields and higher real yield volatilities. Further, we find that increased inflation belief dispersion leads to higher nominal yields and yield volatilities. Our empirical results remain after a series of robustness checks involving changes in specifications and sampling restrictions. We note that the yield volatility results are especially robust. Overall, our findings are consistent with the implications of our model of inflation belief dispersion.

As with any heterogeneous beliefs model, equilibrium asset pricing moments are sensitive to the link between the beliefs of investors and the true data generating measure or the econometrician's beliefs. For instance, Abel (2002) shows that the equity premium puzzle and the risk-free rate puzzle can be solved by assuming that the representative investor is more pessimistic than the econometrician. Similarly, Cecchetti, Lam, and Mark (2000) study various departures from rational expectations and show that asset pricing moments are better matched when consumers are more pessimistic than the econometrician. Recent work by Piazzesi and Schneider (2011) further explores the difference in beliefs between investors and the econometrician regarding the predictability of nominal bond returns. Our results for the level and volatility of yields are robust to the choice of the data generating measure.

Our paper joins a growing literature that explores the role of subjective beliefs or inflation expectations on the term structure. This work includes Ang, Bekaert, and Wei (2007a), Chernov and Mueller (2012), Adrian and Wu (2010), Chun (2011), and Piazzesi and Schneider (2012). Our economy differs from this strand of the literature in that inflation disagreement is explicitly modeled.

Through the speculative trade channel, expected inflation and disagreement about expected inflation impacts the equilibrium wealth distribution in our model. Other work directly appeals to how inflation can impact wealth distributions. Doepke and Schneider (2006) quantitatively explore the impact of inflation on the U.S. wealth distribution under two different assumptions about inflation expectations. Piazzesi and Schneider (2012), using an overlapping generations model with uninsurable nominal risk and disagreement about inflation, study the impact on wealth distributions due to structural shifts in the U.S. economy in the 1970s.

The closest works to ours are Xiong and Yan (2010) and Buraschi and Whelan (2012). Xiong and Yan (2010) build a model with heterogeneous expectations about inflation. They employ logarithmic preferences for which income and substitution effects perfectly offset each other. Therefore, there is no spillover effect of inflation disagreement on real asset prices, including real yields and real yield volatilities. Specifically, Xiong and Yan (2010) theoretically show how heterogeneous expectations drive up nominal yield volatilities and explain the failure of the expectations hypothesis. Buraschi and Whelan (2012), building on the framework of Xiong and Yan (2010), study how disagreement about various macroeconomic quantities, not just inflation, generate predictable variations in excess bond returns. They also use survey data on forecasts to test whether heterogeneous expectations about macroeconomic quantities explain excess bond returns. Our paper differs from Xiong and Yan (2010) and Buraschi and Whelan (2012) in that we derive novel predictions for real and nominal yield levels and real yield volatilities.

2　The Economy

We consider a continuous-time pure exchange economy where otherwise identical investors disagree about expected inflation. The economy has a finite horizon equal to T with a single perishable consumption good. Real prices are measured in units of the consumption good and nominal prices, denoted with a subscript $ throughout, are measured in dollars. Uncertainty is represented by the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$.

The exogenous real aggregate output process $\epsilon(t)$ follows a geometric Brownian motion with dynamics given by

$$d\epsilon(t) = \epsilon(t)\left(\mu_\epsilon\, dt + \sigma_\epsilon\, dz_\epsilon(t)\right), \qquad \epsilon(0) > 0, \tag{2.1}$$

where $z_\epsilon(t)$ is a one-dimensional Brownian motion that represents a real shock.

The exogenous price level $\pi(t)$ converts real prices to nominal prices. For example, nominal aggregate output $\epsilon_\$(t)$ is given by $\epsilon_\$(t) = \pi(t)\epsilon(t)$. The price level has exogenous dynamics given by

$$d\pi(t) = \pi(t)\left(x(t)\, dt + \sigma_{\pi,\epsilon}\, dz_\epsilon(t) + \sigma_{\pi,\$}\, dz_\$(t)\right), \qquad \pi(0) = 1, \tag{2.2}$$

where $x(t)$ denotes expected inflation and $z_\$(t)$ is a one-dimensional Brownian motion that represents a nominal shock. The Brownian motions $z_\epsilon(t)$ and $z_\$(t)$ are uncorrelated. While the price level can be correlated with real aggregate output, this feature is not crucial for our main results. For convenience, we define the total local volatility of the price level as $\sigma_\pi = \sqrt{\sigma_{\pi,\epsilon}^2 + \sigma_{\pi,\$}^2}$ and the local correlation of the price level with the output process as $\frac{d\epsilon(t)}{\sigma_\epsilon \epsilon(t)}\frac{d\pi(t)}{\sigma_\pi \pi(t)} = \rho_{\epsilon\pi}dt$ where $\rho_{\epsilon\pi} = \frac{\sigma_{\pi,\epsilon}}{\sigma_\pi}$.

Expected inflation $x(t)$, while unobservable by investors, follows an Ornstein-Uhlenbeck process[3] with dynamics given by

$$dx(t) = \kappa\left(\bar{x} - x(t)\right)\, dt + \sigma_x\, dz_x(t), \tag{2.3}$$

where $x(0) \sim \mathrm{N}(\bar{x}(0), \sigma_x^2(0))$. The Brownian motion shock to expected inflation $z_x(t)$ is potentially locally correlated with the output process and the price level through $\rho_{x\epsilon}$ and $\rho_{x\pi}$ where $dz_x(t)\frac{d\epsilon(t)}{\sigma_\epsilon \epsilon(t)} = \rho_{x\epsilon}dt$ and $dz_x(t)\frac{d\pi(t)}{\sigma_\pi \pi(t)} = \rho_{x\pi}dt$.

2.1 Beliefs

Investors are heterogeneous in their views on the dynamics of the expected inflation process (2.3). In other words, each investor believes a different model drives expected inflation. While investors still assume that expected inflation is driven by an Ornstein-Uhlenbeck process, they differ with respect to their views on (i) the long run mean of expected inflation $\bar{x}$, (ii) the speed of mean reversion of expected inflation κ, or (iii) both. Having different models for expected inflation may capture, in a reduced form, differing opinions on how a central bank conducts monetary policy through either long-run inflation targeting or aggressiveness in short-run inflation management.

Through different perceived expected inflation dynamics and different initial expected inflation

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[3]The analysis in Section 3 can be adopted to other dynamics for expected inflation such as finite-state Markov processes as in Veronesi (1999, 2000) and David (2008a,b).

priors, investors have heterogeneous beliefs about the current level of expected inflation.[4] For simplicity, investors do not update their views on the model that drives expected inflation. Assuming two types of investors exist, denoted $i = \{1, 2\}$, uncertainty is represented by the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t^{\epsilon, \pi}\}, \mathcal{P}^i\right)$ where $\mathcal{F}_t^{\epsilon, \pi}$ denotes the filtration generated by real output and the price level and $\mathcal{P}^i$ denotes investor i's beliefs. Investors i's best estimate for expected inflation is

$$x^i(t) = \mathrm{E}^i[x(t) \mid \mathcal{F}_t^{\epsilon, \pi}], \quad i \in \{1, 2\}, \tag{2.4}$$

where $\mathrm{E}^i[\cdot]$ denotes the expectation with respect to investor i's belief $\mathcal{P}^i$.

From standard filtering theory, as in Liptser and Shiryaev (1974a,b), investor i's innovation process for the nominal shock $z_\$(t)$ is related to the reference probability $\mathcal{P}$ via

$$dz_\$^i(t) = dz_\$(t) + \frac{x(t) - x^i(t)}{\sigma_{\pi, \$}} dt, \tag{2.5}$$

where $z_\$^i(t)$ is a Brownian motion under investor i's beliefs. There is no disagreement in the real shock $z_\epsilon(t)$ as all investors agree on real output dynamics. Since no investor observes the expected inflation process, they infer it by observing the paths of the price level $\pi(t)$ and real output $\epsilon(t)$. The steady-state solution to this filtering problem is summarized in the next proposition.

Proposition 1. *The price level dynamics under investor i's beliefs are*

$$d\pi(t) = \pi(t) \left(x^i(t) \, dt + \sigma_{\pi, \epsilon} \, dz_\epsilon(t) + \sigma_{\pi, \$} \, dz_\$^i(t)\right), \qquad \pi(0) = 1. \tag{2.6}$$

Investor i's estimate for expected inflation, $x^i(t)$, follows the process:

$$dx^i(t) = \kappa^i \left(\bar{x}^i - x^i(t)\right) dt + \sigma_{x, \epsilon} \, dz_\epsilon(t) + \sigma_{x, \$}^i \, dz_\$^i(t), \tag{2.7}$$

where $x^i(0) \sim N\left(\bar{x}^i(0), \sigma_{x^i(0)}^2\right)$. The volatility $\sigma_{x, \epsilon}$ is common for both investors, that is $\sigma_{x, \epsilon} = \sigma_x \rho_{x\epsilon}$. The volatility $\sigma_{x, \$}^i$, defined in equation (A.2), is a function of κ^i.

Using equation (2.5) as investors agree on the path of the price level, investors' nominal inno-

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[4]Heterogeneous beliefs are modeled with investor-specific priors about these quantities as in for example Detemple and Murthy (1994), Zapatero (1998), and Basak (2000, 2005).

vation processes are linked by

$$dz_\$^2(t) = dz_\$^1(t) - \Delta(t)dt, \qquad \Delta(t) = \frac{x^2(t) - x^1(t)}{\sigma_{\pi,\$}}. \qquad (2.8)$$

The process $\Delta(t)$, termed the *disagreement process* throughout, summarizes current expected infla-
tion disagreement across the two investors. From Proposition 1 and equation (2.8), the dynamics
of the disagreement process are

$$d\Delta(t) = \left(\frac{\kappa^2 \bar{x}^2 - \kappa^1 \bar{x}^1}{\sigma_{\pi,\$}} + \left(\frac{\kappa^1 - \kappa^2}{\sigma_{\pi,\$}} \right) x^1(t) - \left(\kappa^2 + \frac{\sigma_{x,\$}^2}{\sigma_{\pi,\$}} \right) \Delta(t) \right) dt + \frac{\sigma_{x,\$}^2 - \sigma_{x,\$}^1}{\sigma_{\pi,\$}} dz_\$^1(t). \qquad (2.9)$$

This process conveniently captures how disagreement about the dynamics of expected inflation
through the long-run mean $\bar{x}$ and the speed of mean reversion κ guarantees that the investors'
views on expected inflation will not converge. When investors disagree about κ, $\Delta(t)$ is stochastic
as the nominal shock volatility $\sigma_{x,\$}^i$ of investor i's expected inflation estimate is driven by their
κ^i belief. If the only source of disagreement about the dynamics of expected inflation is through
the long-run mean $\bar{x}$, then $\Delta(t)$ is deterministic. For convenience, we will often summarize the
differences between the two investors' models of expected inflation through $\Delta_{\bar{x}} = \bar{x}^2 - \bar{x}^1$ and
$\Delta_\kappa = \kappa^2 - \kappa^1$.

2.2 Security Markets

Trading takes place in a real money market, a risky security, and a nominal money market that
are all in zero net supply. The real money market with price $B(t)$ and the risky security with price
$S(t)$ have posited *real* price dynamics that satisfy

$$dB(t) = B(t)\, r(t)\, dt, \qquad\qquad B(0) = 1, \qquad (2.10)$$

$$dS(t) = S(t)\left(\mu_S(t)\, dt + \sigma_{S,\epsilon}\, dz_\epsilon(t)\right), \qquad\qquad S(0) = 1, \qquad (2.11)$$

where $r(t)$ is the real riskless short rate in the economy, $\mu_S(t)$ the risky security's expected rate
of return, and $\sigma_{S,\epsilon}$ the risky security's volatility. The volatility $\sigma_{S,\epsilon} > 0$ is taken as exogenous
as it defines the risky security since it does not pay any dividends. Here the risky security is
locally perfected correlated with real consumption growth for convenience as the only security

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where expected inflation belief disagreement directly enters return dynamics is the nominal money market.

The nominal money market with price $P_\$(t)$ has posited nominal price dynamics that satisfy

$$dP_\$(t) = P_\$(t)\, r_\$(t)\, dt, \qquad\qquad P_\$(0) = 1, \qquad\qquad (2.12)$$

where $r_\$(t)$ is the nominal riskless short rate in the economy. Applying Itô's lemma to $P(t) = \frac{P_\$(t)}{\pi(t)}$, the real price of the nominal money market, leads to its real price dynamics:

$$dP(t) = P(t)\left(\mu_P(t)\, dt - \sigma_{\pi,\epsilon}\, dz_\epsilon(t) - \sigma_{\pi,\$}\, dz_\$(t)\right), \qquad \mu_P(t) \equiv r_\$(t) - x(t) + \sigma_\pi^2, \qquad (2.13)$$

$$= P(t)\left(\mu_P^i(t)\, dt - \sigma_{\pi,\epsilon}\, dz_\epsilon(t) - \sigma_{\pi,\$}\, dz_\$^i(t)\right), \qquad \mu_P^i(t) \equiv r_\$(t) - x^i(t) + \sigma_\pi^2, \qquad (2.14)$$

where equation (2.14) represents investor i's perceived real price dynamics induced by his expected inflation belief. Since both investors agree on the real price of the nominal money market $P(t)$, the investor-specific expected returns are linked through

$$\mu_P^1(t) - \mu_P^2(t) = \sigma_{\pi,\$}\, \Delta(t) = x^2(t) - x^1(t). \qquad (2.15)$$

This difference in expected returns is solely driven by the disagreement about expected inflation.

All price coefficients $(r(t), r_\$(t), \mu_S(t))$ except the exogenous risky security's volatility $\sigma_{S,\epsilon}$ are determined in equilibrium. This particular security structure is not crucial however. All we require is that each investor faces complete markets. In particular, investors can trade in a claim, the nominal money market, that is exposed to the nominal shock. Without such a security, investors cannot trade on their inflation disagreement.

It is convenient to summarize the price system in terms of investor-specific real stochastic discount factors that capture the investor-specific beliefs, but common Arrow-Debreu prices across investors. Investor i's real stochastic discount factor has dynamics

$$d\xi^i(t) = -\xi^i(t)\left(r(t)\, dt + \theta_\epsilon(t)\, dz_\epsilon(t) + \theta_\$^i(t)\, dz_\$^i(t)\right), \qquad \xi^i(0) \text{ given}, \qquad (2.16)$$

where $\theta_\epsilon(t)$ denotes the market prices of risk of the real shock and $\theta_\$^i(t)$ represents investor i's

perceived market prices of risk to the nominal shock. Specifically,

$$\theta_\epsilon(t) = \frac{\mu_S(t) - r(t)}{\sigma_{S,\epsilon}}, \qquad \theta_\$^i(t) = -\frac{\mu_P^i(t) - r(t)}{\sigma_{\pi,\$}} - \frac{\sigma_{\pi,\epsilon}}{\sigma_{\pi,\$}}\theta_\epsilon(t). \qquad (2.17)$$

Since both investors agree on all security prices, the investor-specific nominal shock market prices of risk are linked through the disagreement process:

$$\theta_\$^2(t) - \theta_\$^1(t) = \Delta(t). \qquad (2.18)$$

2.3 Investor Preferences and Consumption-Portfolio Choice Problem

Investors have external habit preferences similar to Abel (1990, 1999) and Chan and Kogan (2002) defined as

$$U^i = \mathrm{E}^i \left[\int_0^T e^{-\rho t} \frac{1}{1-\gamma} \left(\frac{c^i(t)}{X(t)} \right)^{1-\gamma} dt \right], \qquad i = \{1, 2\}, \qquad (2.19)$$

where ρ denotes the common subjective discount factor and $X(t)$ denotes the standard of living process. The parameter γ measures the local curvature of the utility function, i.e., the relative risk aversion coefficient. We focus on common CRRA-habit preferences across investors to isolate the impact of heterogeneous beliefs from heterogeneous risk aversion on equilibrium prices. Further, the habit-based preference structure helps match quantitative yield curve properties as it is difficult to capture an upward sloping yield curve with CRRA preferences as discussed in Campbell (1986) and Backus and Zin (1994).

The habit level or standard of living, $X(t)$, is measured as a weighted geometric sum of past realizations of aggregate output where $\log(X(t))$ is given by

$$\log(X(t)) = \log(X(0))e^{-\delta t} + \delta \int_0^t e^{-\delta(t-a)} \log(\epsilon(a)) \, da, \qquad \delta > 0, \qquad (2.20)$$

where δ describes the dependence of $X(t)$ on the history of aggregate output.[5] Relative log output

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[5] If δ is large, then shocks to relative output are transitory. The standard of living process then closely resembles current output. If δ is close to zero, then shocks to relative output are persistent. Past aggregate output receives high weight in the standard of living process.

$\omega(t) = \log(\epsilon(t)/X(t))$, a state variable in the model, follows a mean reverting process

$$d\omega(t) = \delta(\bar{\omega} - \omega(t))\,dt + \sigma_\epsilon\,dz_\epsilon(t), \qquad \bar{\omega} = (\mu_\epsilon - \sigma_\epsilon^2/2)/\delta. \tag{2.21}$$

Investor i is endowed with a real endowment stream $\epsilon^i > 0$ where $\epsilon^1(t) + \epsilon^2(t) = \epsilon(t)$. He maximizes utility by choosing a nonnegative consumption process $c^i(t)$ and shares $(\psi_B^i(t), \psi_S^i(t), \psi_P^i(t))$ in the real money market, the risky asset, and the nominal money market respectively subject to the investor's dynamic budget constraint with financial wealth always bounded from below.

Complete markets allow us to use standard martingale techniques (Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989)) to characterize the consumption-portfolio problem of investor i given by

$$\max_{\{c^i(t)\}} \mathrm{E}^i \left[\int_0^T e^{-\rho t} \frac{1}{1-\gamma} \left(\frac{c^i(t)}{X(t)} \right)^{1-\gamma} dt \right]$$

subject to

$$\mathrm{E}^i \left[\int_0^T \xi^i(t) c^i(t)\,dt \right] \leq \mathrm{E}^i \left[\int_0^T \xi^i(t) \epsilon^i(t)\,dt \right]. \tag{2.22}$$

The optimal consumption process $\hat{c}^i(t)$ is

$$\hat{c}^i(t) = e^{-\frac{\rho}{\gamma} t} \left(y_i \xi^i(t) \right)^{-\frac{1}{\gamma}} X(t)^{1-\frac{1}{\gamma}} \tag{2.23}$$

where y^i is the Lagrange multiplier from the static budget constraint given in equation (2.22).

3   Equilibrium Impact of Expected Inflation Disagreement

We characterize the impact of investor disagreement about expected inflation by appealing to general equilibrium restrictions.

Definition 1. *Given preferences, endowments, and beliefs, an equilibrium is a collection of allocations $((c^1(t), \psi_P^1(t), \psi_S^1(t)), (c^2(t), \psi_P^2(t), \psi_S^2(t))$ and a price system $(r(t), \mu_S(t), r_\$(t))$ such that the processes $(c^i(t), \psi_P^i(t), \psi_S^i(t))$ are optimal solutions to investor i's consumption-portfolio choice problem given the perceived price processes in equations (2.10-2.12) and all markets clear. Specifi-*

cally,

$$c^1(t) + c^2(t) = \epsilon(t), \qquad \psi_S^1(t) + \psi_S^2(t) = 0, \qquad \psi_P^1(t) + \psi_P^2(t) = 0, \qquad \forall t \in [0, T].$$

The equilibrium is constructed using a state-dependent representative investor as in Cuoco and He (1994), Basak and Cuoco (1998), and Basak (2000) for example. The state-dependent utility of the representative investor is given by

$$U(\epsilon(t), X(t), \lambda(t)) = \max_{\{c^1(t) + c^2(t) = \epsilon(t)\}} \left(\frac{1}{1-\gamma} \left(\frac{c^1(t)}{X(t)} \right)^{1-\gamma} + \lambda(t) \frac{1}{1-\gamma} \left(\frac{c^2(t)}{X(t)} \right)^{1-\gamma} \right), \qquad (3.1)$$

where $\lambda(t)$ is a nonnegative stochastic welfare weight that captures the impact of heterogeneous beliefs about expected inflation on risk sharing.

Identifying $\lambda(t) = \frac{y^1 \, \xi^1(t)}{y^2 \, \xi^2(t)}$ from the representative investor's optimization, we obtain the following equilibrium characterization of the stochastic welfare weight $\lambda(t)$, the consumption allocations, and the real stochastic discount factors. For convenience, we translate the stochastic welfare weight $\lambda(t)$ into a consumption sharing rule $f(t)$ defined as investor 1's fraction of aggregate consumption.

Proposition 2. *The stochastic welfare weight $\lambda(t)$ has dynamics*

$$d\lambda(t) = \lambda(t)\, \Delta(t) dz_\$^1(t), \qquad (3.2)$$

where $\lambda(0)$ solves either investor's static budget constraint.

The equilibrium consumption allocations are

$$\hat{c}^1(t) = \frac{1}{1 + \lambda(t)^{\frac{1}{\gamma}}} \epsilon(t) = f(t)\, \epsilon(t), \quad \hat{c}^2(t) = \frac{\lambda(t)^{\frac{1}{\gamma}}}{1 + \lambda(t)^{\frac{1}{\gamma}}} \epsilon(t) = (1 - f(t))\, \epsilon(t), \qquad (3.3)$$

where the consumption sharing rule $f(t) \equiv \frac{\hat{c}^1(t)}{\epsilon(t)}$ is given by $f(t) = \frac{1}{1 + \lambda(t)^{\frac{1}{\gamma}}}$.

The investor's equilibrium real stochastic discount factors are

$$\frac{\xi^1(t)}{\xi^1(0)} = e^{-\rho t} \left(\frac{X(t)}{X(0)} \right)^{\gamma-1} \left(\frac{1 + \lambda(t)^{\frac{1}{\gamma}}}{1 + \lambda(0)^{\frac{1}{\gamma}}} \right)^{\gamma} \left(\frac{\epsilon(0)}{\epsilon(t)} \right)^{\gamma} = e^{-\rho t + (1-\gamma)(\omega(t) - \omega(0))} \left(\frac{f(0)}{f(t)} \right)^{\gamma} \frac{\epsilon(0)}{\epsilon(t)}, \quad (3.4)$$

$$\frac{\xi^2(t)}{\xi^2(0)} = \frac{\xi^1(t)}{\xi^1(0)} \frac{\lambda(0)}{\lambda(t)} = \frac{\xi^1(t)}{\xi^1(0)} \left(\frac{f(t)}{1 - f(t)} \frac{1 - f(0)}{f(0)} \right)^{\gamma}. \qquad (3.5)$$

The stochastic weighting process (3.2) summarizes the impact of expected inflation disagreement on equilibrium quantities. Its dynamics are driven by the nominal shock $z_\$^1(t)$ with its volatility determined by the level of disagreement $\Delta(t)$. This leads to each investor's consumption allocation (3.3) to be driven by the nominal shock $z_\$^1(t)$ in addition to the real shock $z_\epsilon(t)$. Given investors choose these nominal-shock impacted consumptions allocations, the equilibrium real stochastic discount factors (3.4)-(3.5) are also impacted due to market clearing. This induces a spillover effect from the nominal side to the real side of the economy.[6] The dynamics of the stochastic discount factors as captured by the equilibrium interest rates and market prices of risk are given in the next proposition.

Proposition 3. *The dynamics of the real and nominal stochastic discount factors, $\xi^i(t)$ and $\xi_\$^i(t) = \frac{\xi^i(t)}{\pi(t)}$, as perceived by investor i are given by*

$$d\xi^i(t) = -\xi^i(t)\left(r(t)dt + \theta_\epsilon(t)dz_\epsilon(t) + \theta_\$^i(t)dz_\$^i(t)\right), \tag{3.6}$$

$$d\xi_\$^i(t) = -\xi_\$^i(t)\left(r_\$(t)dt + \theta_{\$,\epsilon}\,dz_\epsilon(t) + \theta_{\$,\$}^i(t)dz_\$^i(t)\right), \tag{3.7}$$

where $i \in \{1,2\}$. The investor-specific equilibrium real and nominal market prices of risk are

$$\theta_\epsilon(t) = \gamma\,\sigma_\epsilon, \qquad \theta_\$^1(t) = (f(t)-1)\,\Delta(t), \qquad \theta_\$^2(t) = f(t)\,\Delta(t), \tag{3.8}$$

$$\theta_{\$,\epsilon}(t) = \gamma\sigma_\epsilon + \sigma_\pi\rho_{\epsilon\pi}, \qquad \theta_{\$,\$}^1(t) = (f(t)-1)\,\Delta(t) + \sigma_{\pi,\$}, \qquad \theta_{\$,\$}^2(t) = f(t)\,\Delta(t) + \sigma_{\pi,\$}. \tag{3.9}$$

The equilibrium real and nominal interest rates are

$$r(t) = \rho + \mu_\epsilon + \delta(\gamma-1)(\bar\omega - \omega(t)) - \frac{1}{2}(\gamma^2+1)\sigma_\epsilon^2$$
$$+ \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)f(t)(1-f(t))\Delta(t)^2, \tag{3.10}$$

$$r_\$(t) = r(t) + f(t)x^1(t) + (1-f(t))x^2(t) - \gamma\sigma_\epsilon\sigma_\pi\rho_{\epsilon\pi} - \sigma_\pi^2. \tag{3.11}$$

Proposition 3 highlights the impact of speculative trade on the equilibrium stochastic discount factors and hence security prices beyond the standard pricing of the fundamental shock $z_\epsilon(t)$ in

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[6]We have expressed the dynamics for $\lambda(t)$ with respect to investor 1's beliefs for convenience. The dynamics for $\lambda(t)$ under the true beliefs or investor 2's beliefs follow by substituting equation (2.5) or equation (2.8) respectively. As long as $\Delta(t) \neq 0$, time t real quantities are impacted by the nominal shock.

a common belief economy. Although the investors do not disagree about any real quantities, disagreement about expected inflation, a nominal quantity, induces a spillover effect to the real side of the economy as the nominal shock $z_\$(t)$ is now priced in the real stochastic discount factor dynamics. The economic mechanism that drives this feature of the model is similar to a sunspot equilibrium as in Cass and Shell (1983) and Azariadis (1981). In particular, our mechanism is closest to Basak (2000) where investors disagree about a non-fundamental shock and can trade on this disagreement. In our context, disagreement manifests itself through different beliefs about expected inflation.

This pricing of the nominal shock occurs through two channels in each investor's real stochastic discount factor. First, given investors disagree about expected inflation, they disagree on the real expected return on the nominal money market. This disagreement about the risk-return profile of the nominal money market induces speculative trade in it as the investors are willing to take heterogeneous nominal money market positions. Now, the nominal shock $z_\$(t)$ is priced as the investors disagree on the sign of the market price of risk of the nominal shock. With no disagreement $\Delta(t) = 0$, the market price of risk on the nominal shock collapses to zero for both investors. This mechanism, that heterogeneous beliefs about a nominal quantity can induce nominal risks to be priced on the real side of the economy, is distinct from New-Keynesian models as described in Clarida, Galí, and Gertler (1999), Woodford (2003), and Galí (2008) where mechanisms such as sticky prices are imposed so that the nominal side of the economy impacts the real side of the economy.

Second, given the added uncertainty induced by the nominal shock, the equilibrium interest rate is impacted by the demand for extra precautionary savings as captured in the last term of equation (3.10) which is also driven by disagreement $\Delta(t)$. This term is present as long as the investors do not have logarithmic preferences as in Xiong and Yan (2010) for example.

When $\gamma > 1$ ($\gamma < 1$), the equilibrium real interest rate is increasing (decreasing) with disagreement. Intuitively, investors disagree about expected inflation and use the nominal money market to bet against each other. Both investors believe they will capture consumption from the other investor in the future. Classical income and substitution effects then impact the demand for consumption today as discussed in Epstein (1988) and Gallmeyer and Hollifield (2008) for example. Given consumption today is fixed, the real interest rate must adjust to clear markets. If $\gamma = 1$, the

income and substitution effects exactly offset implying no impact on the real interest rate. When $\gamma > 1$, the real interest rate rises to counterbalance increased demand for borrowing. When $\gamma < 1$, the real interest rate falls to counterbalance lowered demand for borrowing.

Investor i's nominal stochastic discount factor dynamics given in equation (3.7) are impacted by expected inflation disagreement in the same way as his real stochastic discount factor dynamics except the equilibrium nominal interest rate given in equation (3.11) is also directly impacted by the consumption share-weighted average of the expected inflation belief across the investors. This extra term arises to convert the real endowment growth μ_ϵ into nominal endowment growth which is driven by the ability of the investors to trade the nominal bond.

Given the impact of expected inflation disagreement on the dynamics of the real and nominal stochastic discount factors, it is natural to ask how inflation disagreement impacts both the real and nominal yield curves. A real bond pays one unit of the consumption good at its maturity and a nominal bond pays one dollar at its maturity. Both are default free and in zero net supply. Hence, the state price densities from Proposition 2 can be used to compute real and nominal bond prices.

At date t, let $B(t; T')$ denote the real price and $B_\$(t; T') = B(t; T')\pi(t)$ the nominal price of a real (inflation-protected) bond maturing at T'. The real price of a real bond with maturity T' is

$$B(t; T') = \mathrm{E}_t^i \left[\frac{\xi^i(T')}{\xi^i(t)} \right]. \tag{3.12}$$

Similarly at date t, let $P(t; T')$ denote the real and $P_\$(t; T') = P(t; T)\pi(t)$ the nominal price of a nominal bond maturing at T'. The nominal price of a nominal bond with maturity T' is

$$P_\$(t; T') = \mathrm{E}_t^i \left[\frac{\xi^i(T')}{\xi^i(t)} \frac{\pi(t)}{\pi(T')} \right] = \mathrm{E}_t^i \left[\frac{\xi_\$^i(T')}{\xi_\$^i(t)} \right]. \tag{3.13}$$

Analogously, define the date t continuously-compounded yield of a real and a nominal zero-coupon bond with maturity T' as $y_B(t; T') = -\frac{1}{T'-t} \log{(B(t; T'))}$ and $y_{P_\$}(t; T') = -\frac{1}{T'-t} \log{(P_\$(t; T'))}$, respectively.

The next proposition summarizes how increased disagreement impacts the level of the real yield curve.

Proposition 4. *Consider two economies with identical consumption allocations at time t. Sup-*

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pose the first economy always exhibits more disagreement across the two investors than the second economy. Disagreement in the first economy, $\bar{\Delta}(s)$, then satisfies $|\bar{\Delta}(s)| \geq |\underline{\Delta}(s)|$ for $s \geq t$ where $\underline{\Delta}(s)$ denotes investor disagreement in the other economy.

Then, real bond prices $\bar{B}(t, T')$ and $\underline{B}(t, T')$ of maturity T' in the higher and lower disagreement economies satisfy

$$
\bar{B}(t, T') \begin{cases} > \underline{B}(t, T') & \text{if } \gamma < 1, \\ = \underline{B}(t, T') & \text{if } \gamma = 1, \\ < \underline{B}(t, T') & \text{if } \gamma > 1. \end{cases} \tag{3.14}
$$

In particular, real bond yields increase with disagreement for $\gamma > 1$ for all t and T'.

The result in Proposition 4 can be generalized if one assumes that there is zero disagreement in the second economy. In this case, one can use Jensen's inequality to prove that the result holds regardless of the dynamics of expected inflation or disagreement as long as they satisfy standard regularity conditions.

Proposition 5. *Consider two economies with identical consumption allocations at time t. Suppose the first economy exhibits expected inflation disagreement, while the second economy does not. Then, the volatility of real bond yields is always higher in the economy with disagreement.*

Note that the Proposition 5 holds in much more general settings. It does not depend on the specific assumption about the dynamics of expected inflation or the dynamics of disagreement as long as agents disagree about expected inflation.

4 Quantitative Impact of Expected Inflation Disagreement

The previous results highlighted qualitative properties of expected inflation disagreement on equilibrium prices. To gain quantitative insights including the impact on nominal yield curves, we assume that the common relative risk aversion across investors γ is an integer. This assumption yields closed-form bond prices as we can construct exact finite expansions of security prices in artificial economies similar to the equilibrium expansions computed in work such as Yan (2008), Dumas, Kurshev, and Uppal (2009), and Bhamra and Uppal (2010).

The following proposition provides a decomposition of the real and nominal stochastic discount

factors that allows for closed-form bond prices. The key to the decomposition is writing the two investor economy stochastic discount factors as sums of single investor habit formation stochastic discount factors with artificial aggregate endowments.

Proposition 6. *When the common risk aversion across the investors γ is an integer, the investor's equilibrium real and nominal stochastic discount factors between times s and t where $s < t$ can be decomposed as*

$$\frac{\xi^i(t)}{\xi^i(s)} = \sum_{k=0}^{\gamma} w_k(s) \frac{\xi_k^i(t)}{\xi_k^i(s)} \qquad and \qquad \frac{\xi_\$^i(t)}{\xi_\$^i(s)} = \sum_{k=0}^{\gamma} w_k(s) \frac{\xi_{k\$}^i(t)}{\xi_{k\$}^i(s)} = \sum_{k=0}^{\gamma} w_k(s) \frac{\xi_k^i(t)}{\xi_k^i(s)} \frac{\pi(s)}{\pi(t)} \tag{4.1}$$

for $i \in \{1,2\}$ where $\xi_k^2(t) = \frac{\xi_k^1(t)}{\lambda(t)}$ and $\xi_k^1(t)$ can be interpreted as a real stochastic discount factor in a fictitious economy given by

$$\xi_k^1(t) = e^{-\rho t} X(t)^{\gamma-1} \lambda(t)^{\frac{k}{\gamma}} \epsilon(t)^{1-2\gamma} = e^{-\rho t + (1-\gamma)\omega(t)} \left(\frac{1 - f(t)}{f(t)} \right)^k \epsilon(t)^{-\gamma}. \tag{4.2}$$

The quantity $w_k(s)$ denotes the weight placed on $\frac{\xi_k^i(t)}{\xi_k^i(s)}$ and is driven by the sharing rule $f(s)$:

$$w_k(s) = \binom{\gamma}{k} \frac{\lambda(s)^{\frac{k}{\gamma}}}{\left(1 + \lambda(s)^{\frac{1}{\gamma}} \right)^\gamma} = \binom{\gamma}{k} f(s)^{\gamma-k} (1 - f(s))^k \tag{4.3}$$

with $\sum_{k=0}^{\gamma} w_k(t) = 1$.

The real stochastic discount factor for economy k given by equation (4.2) can be interpreted as a single investor habit formation economy with a habit given by the process $\omega(t)$ from the original economy and an artificial endowment process of the form $\left(\frac{f(t)}{1-f(t)} \right)^k \epsilon(t)^\gamma$. These state price decompositions allow us to interpret each fictitious economy k as a single investor economy where the difference in beliefs is captured through a fictitious aggregate endowment process. The weighting of each fictitious state price density to recover the actual state price density is solely driven by the sharing rule $f(t)$. The dynamics of the stochastic discount factors in economy k as captured by the market prices of risk and interest rates are given in the next proposition.

Proposition 7. *The dynamics of the real and nominal stochastic discount factors in artificial*

economy k for investor 1 are given by

$$\frac{d\xi_k^1(t)}{\xi_k^1(t)} = -r_k(t)dt - \theta_{k,\epsilon}(t)dz_\epsilon(t) - \theta_{k,\$}^1(t)dz_\$^1(t), \tag{4.4}$$

$$\frac{d\xi_{k\$}^1(t)}{\xi_{k\$}^1(t)} = -r_{k\$}(t)dt - \theta_{k\$,\epsilon}dz_\epsilon(t) - \theta_{k\$,\$}^1 dz_\$^1(t). \tag{4.5}$$

The economy k real and nominal market prices of risk are

$$\theta_{k,\epsilon}(t) = \gamma\sigma_\epsilon, \qquad\qquad \theta_{k,\$}^1(t) = \frac{1}{\sigma_{\pi,\$}}\frac{k}{\gamma}\Delta(t), \tag{4.6}$$

$$\theta_{k\$,\epsilon}(t) = \gamma\sigma_\epsilon + \rho_{\epsilon\pi}\sigma_\pi, \qquad\qquad \theta_{k\$,\$}^1(t) = \frac{1}{\sigma_{\pi,\$}}\frac{k}{\gamma}\Delta(t) + \sigma_{\pi,\$}. \tag{4.7}$$

The economy k real and nominal interest rates are

$$\begin{aligned} r_k(t) &= \rho + \gamma\mu_\epsilon - \delta(\gamma-1)\omega(t) - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2 \\ &\quad - \frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma}-1\right)\frac{1}{\sigma_{\pi,\$}^2}(\Delta(t))^2, \end{aligned} \tag{4.8}$$

$$r_{k\$}(t) = r_k(t) + \left(1 - \frac{k}{\gamma}\right)x^1(t) + \frac{k}{\gamma}x^2(t) - \gamma\rho_{\epsilon\pi}\sigma_\pi\sigma_\epsilon - \sigma_\pi^2. \tag{4.9}$$

Proposition 7 highlights the benefit of decomposing the stochastic discount factors as weighted averages of the stochastic discount factors in artificial economies. From equations (4.6)-(4.9), the economy k market prices of risk and interest rates are independent of the sharing rule $f(t)$ and are only driven by investors' beliefs about expected inflation. Given beliefs evolve exogenously, the economy k price system is solely driven by the exogenous state variables $(\omega(t), x^1(t), x^2(t))$. The impact of the sharing rule $f(t)$ re-enters the equilibrium stochastic discount factors through the weights given by equation (4.3).

By decoupling the sharing rule from the artificial economies' price dynamics, we can express real and nominal bond prices as sums of real and nominal bond prices in the artificial economies. Specifically, the real price of a real zero-coupon bond is

$$B(t;T') = \sum_{k=0}^{\gamma} w_k(t)B_k(t;T'), \tag{4.10}$$

where $B_k(t; T')$ denotes the real price of a zero-coupon real bond in artificial economy k given by

$$B_k(t; T') = E_t^1 \left[\frac{\xi_k^1(T')}{\xi_k^1(t)} \right] = E^1 \left[\frac{\xi_k^1(T')}{\xi_k^1(t)} \,\middle|\, \omega(t) = \omega, x^1(t) = x^1, x^2(t) = x^2 \right]. \tag{4.11}$$

Likewise, the nominal price of a nominal zero-coupon bond is

$$P_\$(t; T') = \sum_{k=0}^{\gamma} w_k(t) P_{\$k}(t; T'), \tag{4.12}$$

where $P_{k\$}(t; T')$ denotes the nominal price of a zero-coupon nominal bond in artificial economy k given by

$$P_{k\$}(t; T') = E_t^1 \left[\frac{\xi_{k\$}^1(T')}{\xi_{k\$}^1(t)} \right] = E^1 \left[\frac{\xi_{k\$}^1(T')}{\xi_{k\$}^1(t)} \,\middle|\, \omega(t) = \omega, x^1(t) = x^1, x^2(t) = x^2 \right]. \tag{4.13}$$

From the structure of the artificial economies, we now show that the artificial real and nominal term structures are in the class of quadratic Gaussian term structure models as studied in Ahn, Dittmar, and Gallant (2002) and Leippold and Wu (2002). To show this mapping, we largely adopt the same notation as Ahn, Dittmar, and Gallant (2002) for the state vector $Y(t)$ in the economy, where $Y(t) = (x^1(t), x^2(t), \omega(t))'$. Additional details are given in the Appendix.

Real and nominal bond prices in artificial economy k follow a quadratic Gaussian term structure model and are summarized in the following proposition.

Proposition 8. *The real and nominal bond prices, $B_k(t; T')$ and $P_{k\$}(t; T')$, in artificial economy k are exponential quadratic functions of the state vector given by*

$$B_k(t; T') = exp \left\{ \mathcal{A}_k(T' - t) + \mathcal{B}_k(T' - t)'Y(t) + Y(t)'\mathcal{C}_k(T' - t)Y(t) \right\}, \tag{4.14}$$

$$P_{k\$}(t; T') = exp \left\{ \mathcal{A}_{k\$}(T' - t) + \mathcal{B}_{k\$}(T' - t)'Y(t) + Y(t)'\mathcal{C}_{k\$}(T' - t)Y(t) \right\}, \tag{4.15}$$

where the coefficients $(\mathcal{A}_k(\cdot), \mathcal{B}_k(\cdot), \mathcal{C}_k(\cdot), \mathcal{A}_{k\$}(\cdot), \mathcal{B}_{k\$}(\cdot), \mathcal{C}_{k\$}(\cdot))$ are solutions to ordinary differential equations summarized in the Appendix.

When the investors exhibit no disagreement in the speed of mean reversion of expected inflation κ, $\mathcal{C}_k(\cdot) = \mathcal{C}_{k\$}(\cdot) = 0$ implying the bond prices in artificial economy k fall into the class of Gaussian

affine term structure models.

Summarizing, when both investors are endowed with an integer risk aversion γ, real and nominal bond prices can be expressed as expansions of artificial economics with quadratic Gaussian term structures. Disagreement in the speed of mean reversion κ drives the quadratic term. Without κ disagreement, the term structures in artificial economy k follow affine term structure models. The weights in the expansions are driven by the sharing rule $f(t)$ providing an additional channel to impact bond prices and their dynamics.

4.1   Numerical Examples

We now explore how real and nominal yield curve properties are quantitatively impacted by expected inflation disagreement in a numerical example. Our parameters chosen are meant to be broadly consistent with past work such as Campbell and Cochrane (1999), Chan and Kogan (2002), and Wachter (2006). For the expected inflation process, we assume that $\bar{x} = 3\%$ and $\kappa = 0.3$ which are quantities consistent with the Kalman filter-based estimations in Munk et al. (2004). To use the closed-form bond pricing expressions, investors have a common integer risk aversion coefficient of $\gamma = 7$. Table 1 summarizes the other parameters used.

Different beliefs about the model driving expected inflation are captured both through an investors-specific long run mean $\bar{x}^i$ and speed of mean reversion κ^i in our examples. We use the true parameters $\bar{x} = 3\%$ and $\kappa = 0.3$ to anchor the investors' beliefs by requiring the "average" beliefs across investors to equal the true parameters. Specifically, $\bar{x}^1 = 2.5\%$ and $\bar{x}^2 = 3.5\%$ and hence the difference in beliefs about the long run mean is $\Delta_{\bar{x}} \equiv \bar{x}^2 - \bar{x}^1 = 0.01$. The difference in the speed of mean reversion is $\Delta_{\kappa} \equiv \kappa^2 - \kappa^1 = 0.15 - 0.45 = -0.3$. Both investors use real output and the price level to update expected inflation estimates. Steady state expected inflation estimates are 2.79% for investor 1 and 3.09% for investor 2, respectively; this corresponds to a steady state difference of 30 basis points.

Quantitative Impact of Belief Disagreement

We first show that expected inflation disagreement can have a significant impact on yield curves by exploring some simulated yield curve properties in Figure 1. Average yield levels and average yield

volatilities are presented for both nominal and real yield curves with maturities from 0 to 5 years. In addition to our expected inflation disagreement model (denoted "Disagreement" with a dashed line), we also plot several single investor benchmark models. These include learning economies fully populated by investor $i \in \{1, 2\}$ with his corresponding beliefs (denoted "Common Beliefs - Investor i" with a dash-dotted (dotted) line for investor 1 (2)), a learning economy populated by an investor with the correct beliefs about the expected inflation model (denoted "Common Beliefs - Correct"), and a full information economy (denoted "Full Info"). For real yield curve properties, the common belief learning economies and the full information economy are identical as only nominal yield curve properties are impacted. To compare against empirical yield curve properties, we also plot the corresponding average yield levels and average yield volatilities using U.S. Treasury bond data from 1981 to 2011 (denoted "Data" with a solid line).[7] The yield curve properties from the models are determined by simulating 2000 paths of monthly yields for 31 years to match the same time series length as our data.

From Figure 1, we see that expected inflation disagreement has a large impact on average yield levels (top plots) and average yield volatilities (bottom plots) for both nominal (left plots) and real (right plots) yield curves. The impact of the precautionary savings motive induced by belief disagreement is especially stark for the average real yield curve level by comparing the disagreement economy with the full information economy. While the expected inflation steady-state difference across investors is only 30 basis points, it translates into shifting the yield curve up by approximately 150 basis points. It also has a considerable impact on average real yield volatilities as they rise by approximately 22% under the disagreement economy compared to the full information economy at the short end of the yield curve. This additional volatility arises through investor disagreement about κ as $\Delta(t)$ is then stochastic.

The nominal yield curve properties are similarly impacted by inflation disagreement. However, the impact is even stronger as nominal interest rates are now directly impacted by inflation disagreement in addition to the precautionary savings motive that also impacts real interest rates. In the top left plot of Figure 1, the disagreement economy's nominal yield curve is shifted up by over 200 basis points relative to the full information economy's yield curve. From the bottom left plot, yield volatilities rise by approximately 28% for the disagreement economy relative to the full

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[7]Section 5 describes the specifics of the data used.

information economy.

The figure also highlights the importance of disagreement and not just learning as the common belief learning economies exhibit a much smaller increase in average yield levels and volatilities. The disagreement economy nominal yield curve properties are always above all of the common belief economies by a significant amount. Additionally, the learning economies exhibit no spillover effect to the real yield curve.

Overall, the addition of expected inflation disagreement leads to yield curves that fit the data reasonably well. In particular, expected inflation disagreement helps fit bond yield volatilities relative to learning economies.

Impact of Varying Disagreement and the Sharing Rule

Moving beyond average yield curve properties, we explore how two key state variables, disagreement $\Delta(t)$ and the sharing rule $f(t)$, impact conditional yield curve properties. Figure 2 plots yield curve properties conditional on initial disagreement $\Delta(0)$, while Figure 3 plots yield curve properties conditional on the initial sharing rule $f(0)$.

In Figure 2, three different levels of initial disagreement are considered. The "steady-state disagreement" case corresponds to setting $\Delta(0)$ equal to its steady state value of $\frac{0.0309-0.0279}{\sigma_{\pi,\$}}$. The "high disagreement" case sets $\Delta(0)$ equal to a one standard deviation increase in the steady-state disagreement Δ, a value of $\frac{0.0336-0.0253}{\sigma_{\pi,\$}}$. Finally, the "zero disagreement" case corresponds to a disagreement economy where the two investors initially do not disagree about expected inflation $\Delta(0) = 0$, but still have different models of expected inflation. This implies non-zero disagreement in the future. All other parameters used are as in Table 1.

Figure 2 demonstrates that the current amount of disagreement has a significant impact on yield curves. When disagreement goes down, yields go down, yield curves steepen, and yield volatilities fall. When disagreement goes up, nominal yields rise and the slopes of the yield curves flatten. In this particular example, the yield curves even invert. Additionally, yield volatilities increase as disagreement increases. While disagreement increases bond yield volatility, it also flattens the average yield curve. Based on our numerical work, the feature that increased disagreement increases nominal average yields and volatilities as well flattens the yield curve seems robust across a large parameter space. So, our analytic results concerning how disagreement impacts real yield levels

and volatilities carry over to nominal yield curves numerically.

Quantitatively, the example also highlights that changes in inflation disagreement can have a large impact on real and nominal yield curves. For example, the nominal yield curve at a maturity of one year shifts by over 150 basis points when moving between the no disagreement and the high disagreement case. At longer maturities, the impact is smaller. At a five year maturity, the shift is approximately 50 basis points. If expected inflation is very persistent ($\kappa \approx 0$), then an increase in disagreement has the same effect on short term and long term yields. However, if expected inflation shows little persistence with κ large, then disagreement has almost no effect on long term yields.

Figure 3 sets the initial disagreement $\Delta(0)$ to the steady state level and show the impact of inflation disagreement on the nominal and real yield curves when the consumption share $f(0)$ is varied. The left plot shows the nominal yield curve. The impact of disagreement is higher when the two investors have similar consumption shares. When the economy is dominated by one of the two investors, the impact of disagreement approaches zero as the economy is converging to a single investor learning setting. The real yields show a similar pattern, although the maximums for nominal yields are somewhat closer to zero than for the real yields. This follows from the Fisher effect: as the second investor believes inflation to be higher than the first investor, yields are more influenced by his view when his consumption share is larger.

Effect of Inflation Expectations on Real Yields

Given disagreement about expected inflation impacts real yields in our model, it also provides a channel for the level of inflation expectations to impact real yield curves. To see this, it is useful to revisit the dynamics of the disagreement process $\Delta(t)$ given by (2.9). Here we see the drift of the disagreement process is not only driven by its current level, but also driven by investor 1's belief about expected inflation $x^1(t)$ if investors disagree about κ. So, investor 1's expected inflation belief $x^1(t)$ can help predict future levels of $\Delta(t)$.

While Figure 2 shows the quantitative impact of current disagreement on real yield curves, Figure 4 demonstrates the effect of investor 1's current inflation expectation x^1 on the real yield curve for a given amount of current disagreement Δ. The real short rate and real bond yields with maturities ranging from one to five years are plotted. Each of the four plots considers different configurations of the expected inflation model as perceived by the two investors. This again is

summarized through differences in long-run means $\Delta_{\bar{x}} = \bar{x}^2 - \bar{x}^1$ and differences in the speed of mean reversion $\Delta_\kappa = \kappa^2 - \kappa^1$. In each plot, the level of disagreement Δ is set equal to its long-run mean.

Expected inflation has no effect on the real short rate in any expected inflation model configuration, but it still impacts real bond yields as long as there is disagreement about κ. If $\Delta_\kappa = 0$ (top left plot of Figure 4), then the disagreement process is deterministic. Expected inflation does not predict future disagreement; it hence does not predict future consumption shares.

However, if there is disagreement about κ, then expected inflation predicts future disagreement. Therefore it affects real yield levels. For instance, the top right plot of Figure 4 shows real yields if there is no disagreement about the long-run mean ($\Delta_{\bar{x}} = 0$). Here both investors agree on the current expected inflation rate and the long-run mean of 3%. However, they disagree on the speed of mean reversion. Therefore, a current inflation expectation that deviates from its long-run mean predicts high future disagreement. This leads to a U-shaped relation between expected inflation and real yields with minimums at the common long run mean of 3%. If investors disagree about the long-run mean and the speed of mean reversion (bottom row of Figure 4), then the relation is still U-shaped, but the minimums shift upward (downward) if the disagreement about the long-run mean and the speed of mean reversion has the same (opposite) sign.

5   Empirical Evidence

Given the model's linkage between expected inflation disagreement and yield curve properties, we empirically explore these relationships using U.S. Treasury bond data. We focus on our model's predictions on how expected inflation disagreement impacts real and nominal yield curve levels and volatilities.

Our nominal yield sample is taken from the Fama-Bliss Discount Bond File where we use 1 to 5 year nominal yields extracted from artificial discount bonds. To match the availability of our measure of expected inflation disagreement described below, our nominal yield data is monthly in frequency and ranges from January 1978 to December 2011. Building a long time series of U.S. real bond yields is problematic as the U.S. Treasury only began issuing Treasury Inflation Protected Securities (TIPS) in 1997. To address this issue, we construct two different data sets of real yields.

First, we merge the quarterly implied real yields in Chernov and Mueller (2012) (Q3 1971 to Q4 2002) with the TIPS data in Gürkaynak, Sack, and Wright (2010) giving us quarterly real yields from Q1 1978 to Q4 2011 to match our expected inflation disagreement measure.[8] Second, we employ a common approximation of real yields by subtracting realized inflation (the CPI) over the bond's maturity from the bond's nominal yield using our data from the Fama-Bliss Discount Bond File giving us data that is monthly in frequency and ranges from January 1978 to December 2011. From these nominal and real yield series, we estimate a GARCH(1,1) for yield volatilities.[9]

5.1 Measure of Expected Inflation Disagreement - Dispersion

To build a measure of disagreement from expected inflation forecasts, we use three commonly used surveys of economic data — the Livingston Survey, the Michigan Surveys of Consumers, and the Survey of Professional Forecasters. From the raw series of forecasts, we compute the mean forecast ("Mean Inflation") as well as the standard deviation around the mean forecast, which we call "Dispersion," at each point in time. Dispersion is a proxy for inflation disagreement in our model. The Michigan Surveys of Consumers data is monthly (available since January 1978), the Survey of Professional Forecasters data is quarterly (available since September 1981), and the Livingston Survey data is semiannual (available since December 1946). Below we focus on results obtained from the Michigan Surveys of Consumers data given the data is available at the highest frequency. Results from using the other surveys are available from the authors.

Figure 5 shows the evolution of mean beliefs and dispersion for the Michigan Surveys of Consumers over time with NBER recessions as gray shaded areas. In the top panel, the monthly one year ahead CPI is plotted in addition to the mean survey forecast. We see from the figure that the mean of the survey forecast appears to contain valuable information regarding future realizations of inflation. In other words, the mean inflation forecast predicts realized inflation as in Ang, Bekaert, and Wei (2007a). Yet, we also see from the figure that consumers at times are surprised by low realizations of inflation. Nonetheless, we do not find that professional forecasters outperform consumers. Specifically, in monthly predictive regressions, we find the following Newey-West corrected

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[8]See https://www.federalreserve.gov/econresdata/researchdata.htm for the Gürkaynak, Sack, and Wright (2007, 2010) data.

[9]In unreported results, we expand the maturities of yields considered using the CRSP Risk-Free Rates File (1 and 3 month yields) and the data set from Gürkaynak, Sack, and Wright (2007) (6 to 30 year yields). The results are similar and are available from the authors.

t-statistics on Mean Inflation 2.85, 12.81, and 2.78 for the Livingston Survey, the Michigan Surveys of Consumers, and the Survey of Professional Forecasters, respectively. The adjusted R^2 of these regressions are 19.3, 28.3, and 3.0. Following a RMSE evaluation criterion to discriminate on predictive ability as in Ang et al. (2007a) for example, we find that the mean forecast of each survey shows predictive power for inflation and that performance is similar across surveys. These predictive regressions show similar relative performance at quarterly, semi-annual, and annual horizons and appear insensitive to alternative specifications including replacing the mean forecast with the median.

From the bottom panel in Figure 5, we note a rather high inflation belief dispersion derived from the Michigan Surveys of Consumers. The high dispersion embedded in the data of the Michigan Surveys of Consumers, also relative to other surveys such as the Livingston Survey and the Survey of Professional Forecasters, need not necessarily be surprising, considering that the Michigan Survey asks questions about price changes from the perspective of households. Since households have arguably different consumption bundles, this probably implies increased dispersion relative to a hypothetical survey that asks questions about the CPI instead. In addition, Malmendier and Nagel (2011) argue that dispersion in consumer forecast data is higher than in the professional forecaster data as older consumers consistently overestimate both inflation and its volatility based on past experience. Figure 5 supports this view as the path of the mean inflation forecast almost always lies above realized inflation after the high inflation period at the beginning of our sample.

5.2   Belief Dispersion Regressions

Our main prediction is that inflation belief dispersion drives up real yields and yield volatilities which we now test through regression analysis. Unfortunately, the TIPS time series it too short to be of great use for our purpose.[10] Instead, we employ the quarterly data from Chernov and Mueller (2012) together with the TIPS data to test our prediction.[11] Table 2 present coefficient estimates from these regressions for the following maturities: 1-year, 2-year, 3-year, and 5-year.

---

[10]We only have 96 monthly observations using the TIPS data. Yield regressions show the correct sign, but all coefficients are insignificant. Yield volatility regressions always show the correct sign and have highly statistically significant coefficient estimates up to the 20-year maturity.

[11]In this specification, we use TIPS data from 2004 on for 2-year and 3-year maturities and interpolate with cubic splines the rates for 2003. For the 5-year maturity, we use TIPS data from 2003 onwards. The 4-year data is not available in Chernov and Mueller (2012).

Panel A presents results for our full sample, while Panel B presents results excluding the high inflationary period early in the sample. For each maturity, the table presents two models. Model 1 contains a constant and Dispersion as explanatory variables. Model 2 includes Mean Inflation as an additional explanatory variable to control for feedback effects as in Section 4.1. We see that in the real yield level regressions in Panel A the sign of the coefficient, for Model 1 regressions, is always positive consistent with our prediction. The Newey-West corrected t-statistics are significant for 2-year and 3-year maturities. Model 2 shows similar pattern with significant Dispersion coefficients starting from 2-year maturity. The real yield volatility regressions of Table 2 present estimates of the dispersion regressions for yield volatilities for Models 1 and 2. We see that all coefficient estimates for Dispersion are significant and positive over both time periods.

To address concerns regarding the small number of observations, we supplement the above results with imputed real yields computed from nominal yields and realized inflation over the inflation forecast horizon. Table 3 presents coefficient estimates from the two regression models for the following maturities: 1-year, 2-year, 3-year, 4-year, and 5-year. We see that in the real yield level regressions, the sign of the coefficient is always positive consistent with our prediction using the Full Sample in Panel A. For Model 1, the Newey-West corrected t-statistics are significant at the short end of the yield curve, until the 3-year yield, consistent with our assertion that imputed real yields should work well as long as the variation in the inflation risk premium is small. The Newey-West corrected t-statistics are significant for all maturities for Model 2 and for the No High Inflation Period Sample in Panel B. Additionally, all regressions with yield volatilities as the dependent variable show the expected sign with highly significant coefficient estimates.

Next we ask whether disagreement about expected inflation between investors, as measured by inflation forecast dispersion, shows a significantly positive relation with nominal bond yields in regression models. Table 4 presents estimates of these dispersion regression coefficients for the following maturities: 1-year, 2-year, 3-year, 4-year, and 5-year. For each maturity, the table presents the same two models as in the real yield regressions. Model 1 contains a constant and Dispersion as explanatory variables. We recognize that periods with high inflation imply high dispersion. At least for nominal yields, such a mechanical relation can lead to a rejection of the null hypothesis simply because high dispersion implies high inflation which obviously implies higher nominal bond yields. Model 2 addresses this concern by including Mean Inflation as an explanatory variable.

Mean Inflation is also a proxy, albeit a rough one, for the consumption-weighted mean inflation forecast, which is a state variable in our model. All coefficients of Dispersion in the nominal yield level regressions have positive signs and are highly statistically significant across both samples. We then ask if Dispersion shows a significantly positive relation with nominal bond yield volatilities. The nominal yield volatility regressions of Table 4 presents estimates of these dispersion regression coefficients. Again, the table presents Model 1 and Model 2 for each maturity. As for yields, all coefficients of Dispersion in the nominal yield volatility regressions of Table 4 have positive signs and are highly statistically significant.

5.3   Robustness

Several robustness checks, including tests using the Livingston Survey and the Survey of Professional Forecasters, are available from the authors.[12] Two robustness checks related to our model and one check related to the persistence of the series are worth mentioning however. First, the model contains one more state variable, namely relative log output ω. Since ω is uncorrelated with dispersion and the consumption-weighted inflation forecast in our theoretical model, both regression models presented earlier should remain well-specified regressions. Nonetheless, when we include the log price/dividend ratio of the US stock market as proxy for ω, we find that the regression results in Table 3 and Table 4 are unchanged and that the log price/dividend ratio shows the expected negative sign. Second, one theoretical concern with our regressions is that if dispersion is correlated with inflation volatility, it might be priced which is contrary to our model. However, our regression results are robust to the inclusion of inflation volatility estimated through a GARCH(1,1).

In addition, all our time series exhibit high persistence, so we include lagged Dispersion in a series of robustness checks of our regressions. The sign of the coefficient estimates on Dispersion in these regressions remain the same and the coefficients are still significant. From our main regressions and robustness tests, we find that the empirical results are largely statistically significant, robust, and consistent with the model's theoretical predictions.

---

[12]The yield regressions using the Livingston Survey and the Survey of Professional Forecasters show the correct signs for Dispersion but result frequently in insignificant coefficient estimates, possibly due to the low frequency of the data and a limited number of observations. Yield volatility regressions, however, always show positive and significant coefficient estimates for Dispersion.

6 Conclusion

We study how differences in beliefs about expected inflation affect the real and nominal term structures. Our model shows that differences in beliefs about expected inflation impact the equilibrium real pricing kernel generating a spill-over effect from the nominal to the real side of the economy. We find that heterogeneous beliefs about expected inflation have a strong impact on the level and volatility of the real yield curve. When both investors share common preferences over consumption with a coefficient of relative risk aversion greater than one, real average yields across all maturities rise as disagreement increases. When the coefficient of relative risk aversion is less than one, real average yields fall as disagreement increases. Over both cases, yield volatilities increase with disagreement. For additional nominal term structure implications, we consider a simplifying case where the term structures can be computed in closed-form as a consumption-weighted quadratic Gaussian term structure model. We demonstrate numerically how the nature of the difference in beliefs about inflation among investors is important in generating features of the real and nominal yield curves. From empirical work, we find support for our model's predictions about the relation between inflation disagreement and yield curve properties for both real and nominal term structures.

Appendix

Proofs and Auxiliary Results

Proof of Proposition 1. The result follows from Theorem 12.1 of Liptser and Shiryaev (1974b). The volatilities $\sigma_{x,\epsilon}^i$ and $\sigma_{x,\$}^i$ for $i = 0, 1, 2$ are

$$\sigma_{x,\epsilon}^i = \sigma_{x,\epsilon} = \sigma_x \, \rho_{x\epsilon}, \tag{A.1}$$

$$\sigma_{x,\$}^i = \frac{\sigma_x}{\sqrt{1 - \rho_{\epsilon\pi}^2}} \left(\rho_{x\pi} - \rho_{\epsilon\pi}\rho_{x\epsilon} + \frac{1}{\sigma_\pi \sigma_x} v^i \right) = \frac{\sigma_x}{\sqrt{1 - \rho_{\epsilon\pi}^2}} \left(\rho_{x\pi} - \rho_{\epsilon\pi}\rho_{x\epsilon} \right) + \frac{v^i}{\sigma_{\pi,\$}}, \tag{A.2}$$

where v^i is investor i's estimation error.

Suppose the estimation error v^i is equal to its steady state value, i.e., it is a constant, given by

$$a \left(v^i \right)^2 + b^i v^i + c = 0, \tag{A.3}$$

with

$$a = -\frac{1}{\left(1 - \rho_{\epsilon\pi}^2 \right) \sigma_\pi^2}, \tag{A.4}$$

$$b^i = -2\kappa^i - \frac{2\sigma_x}{\sigma_\pi \left(1 - \rho_{\epsilon\pi}^2 \right)} \left(\rho_{x\pi} - \rho_{\epsilon\pi}\rho_{x\epsilon} \right), \tag{A.5}$$

$$c = \frac{\sigma_x^2}{1 - \rho_{\epsilon\pi}^2} \left(1 - \rho_{\pi\epsilon}^2 - \rho_{x\pi}^2 - \rho_{x\epsilon}^2 + 2\rho_{\pi\epsilon}\rho_{x\pi}\rho_{x\epsilon} \right). \tag{A.6}$$

$\square$

Proof of Proposition 2. The proof follows from Karatzas et al. (1990) with the appropriate modifications taken to accomodate for investors facing different state prices through heterogeneous beliefs.

$\square$

Proof of Propositions 3. The proof follows from applying Itô's lemma to each investor's first order conditions, imposing market clearing, and match coefficients in the dynamics of the real and nominal state price densities.

$\square$

Proof of Proposition 4. The real bond price written in terms of investor 1's beliefs is given by

$$
\begin{aligned}
B(t;T') &= \mathrm{E}_t^1 \left[\frac{\xi^i(T')}{\xi^i(t)} \right] \\
&= e^{-\rho(T-t)} \mathrm{E}_t^1 \left[\frac{\left(1 + \lambda(T)^{\frac{1}{\gamma}} \right)^\gamma}{\left(1 + \lambda(t)^{\frac{1}{\gamma}} \right)^\gamma} \left(\frac{\epsilon(T)}{\epsilon(t)(1 + \lambda(T)^{\frac{1}{\gamma}})^\gamma} \right)^{-\gamma} e^{(1-\gamma)(\omega(T) - \omega(t))} \right].
\end{aligned}
$$

Given we only focus on differences in beliefs about inflation, $\lambda(t)$ and $\epsilon(t)$ are uncorrelated

implying

$$B(t;T') = e^{-\rho(T-t)}\mathrm{E}_t^1\left[\left(\frac{\epsilon(T)}{\epsilon(t)}\right)^{-\gamma}e^{(1-\gamma)(\omega(T)-\omega(t))}\right] \times \mathrm{E}_t^1\left[\frac{\left(1+\lambda(T)^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(1+\lambda(t)^{\frac{1}{\gamma}}\right)^{\gamma}}\right].$$

Increasing disagreement only impacts the last expectation. First, note that real bond prices are more volatile under disagreement as under the benchmark of no disagreement, $\lambda(t)$ is a constant.

To establish how increased disagreement impacts the expectation given by

$$\mathrm{E}_t^1\left[\frac{\left(1+\lambda(T)^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(1+\lambda(t)^{\frac{1}{\gamma}}\right)^{\gamma}}\right],$$

we can apply the comparison theorem stated below due to Hajek (1985) where the weighting process $\lambda(t)$ is a martingale. □

Theorem 1 (Mean Comparison Theorem Adapted from Hajek (1985)). *Let x be a continuous martingale with representation*

$$x(t) = x(0) + \int_0^t \sigma(s)dw(s)$$

such that for some Lipschitz continuous function ρ, $|\sigma(s)| \leq \rho(x(s))$ and let y be the unique solution to the stochastic differential equation

$$y(t) = x(0) + \int_0^t \rho(y(s))dw(s).$$

Then, for any convex function Φ and any $t \geq 0$,

$$E\left[\Phi(x(t))\right] \leq E\left[\Phi(y(t))\right].$$

Proof of Proposition 5. The proof follows from noting that the real bond price can be expressed as

$$B(t;T') = \underbrace{e^{-\rho(T-t)}\mathrm{E}_t^1\left[\left(\frac{\epsilon(T)}{\epsilon(t)}\right)^{-\gamma}e^{(1-\gamma)(\omega(T)-\omega(t))}\right]}_{\text{real bond price without disagreement}} \times \underbrace{\mathrm{E}_t^1\left[\frac{\left(1+\lambda(T)^{\frac{1}{\gamma}}\right)^{\gamma}}{\left(1+\lambda(t)^{\frac{1}{\gamma}}\right)^{\gamma}}\right]}_{\text{effect from disagreement}}.$$

As the effect from disagreement is stochastic and independent of the real bond price in the case of no disagreement, it follows that the volatility of real bond prices, and hence real yields, are more volatile with disagreement. □

Proof of Proposition 6. The proof follows by applying Proposition 3. □

Proof of Proposition 8. The proof directly follows from applying Proposition 9 in the Appendix. Mapping into the Ahn et al. (2002) setting, the dynamics of $Y(t) = (x^0(t), x^1(t), x^2(t), \omega(t))'$ are

$$dY(t) = (\mu + \xi Y(t))\ dt + \Sigma dZ_2(t), \tag{A.7}$$

where

$$\mu = \left(\kappa_0 \bar{x}_0, \kappa_1 \bar{x}_1, \kappa_2 \bar{x}_2, \delta \bar{\omega}\right)' \in \mathcal{R}^4, \tag{A.8}$$

$$\xi = \begin{pmatrix} -\kappa_0 & 0 & 0 & 0 \\ \frac{\sigma_{x,\$}^1}{\sigma_{\pi,\$}} & -\left(\kappa^1 + \frac{\sigma_{x,\$}^1}{\sigma_{\pi,\$}}\right) & 0 & 0 \\ \frac{\sigma_{x,\$}^2}{\sigma_{\pi,\$}} & 0 & -\left(\kappa^2 + \frac{\sigma_{x,\$}^2}{\sigma_{\pi,\$}}\right) & 0 \\ 0 & 0 & 0 & -\delta \end{pmatrix} \in \mathcal{R}^{4\times4}, \tag{A.9}$$

$$\Sigma = \begin{pmatrix} \sigma_{x,\epsilon}^0 & \sigma_{x,\$}^0 \\ \sigma_{x,\epsilon}^1 & \sigma_{x,\$}^1 \\ \sigma_{x,\epsilon}^2 & \sigma_{x,\$}^2 \\ \sigma_\epsilon & 0 \end{pmatrix} \in \mathcal{R}^{4\times2}, \tag{A.10}$$

and

$$Z_2(t) = \left(z_{\epsilon,0}(t), z_\$^0(t)\right)' \in \mathcal{R}^2. \tag{A.11}$$

The volatilities $\sigma_{x,\epsilon}^i$ and $\sigma_{x,\$}^i$ for $i = 0, 1, 2$ are given in Proposition 1.

The coefficients for the real bond price, $\mathcal{A}_k(T'-t)$, $\mathcal{B}_k(\tau)$, and $\mathcal{C}_k(T'-t)$, are the solutions of the ordinary differential equations given in Proposition 9 where

$$\eta_{0,k} = -\left(\gamma\sigma_\epsilon, 0\right)' \tag{A.12}$$

$$\eta_{Y1,k} = 0_4 \tag{A.13}$$

$$\eta_{Y2,k} = -\frac{1}{\sigma_{\pi,\$}}\left(1, -\left(1 - \frac{k}{\gamma}\right), -\frac{k}{\gamma}, 0\right)' \tag{A.14}$$

$$\alpha_k = \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2 \tag{A.15}$$

$$\beta_k = (0, 0, 0, \delta(1-\gamma))' \tag{A.16}$$

$$\Psi_k = -\frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma} - 1\right)\frac{1}{\sigma_{\pi,\$}^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.17}$$

The matrix Ψ_k is positive semidefinite because $k/\gamma \leq 1$. Note Ψ_k is singular and that $\psi_K = 0_{4\times4}$ if $k = 0$ or $k = \gamma$.

The coefficients for the nominal bond price, $\mathcal{A}_{k\$}(T'-t)$, $\mathcal{B}_{k\$}(\tau)$, and $\mathcal{C}_{k\$}(T'-t)$, are the solutions

of the ordinary differential equations given in Proposition 9 where

$$\eta_{0,k\$} = -\left(\gamma\sigma_\epsilon + \rho_{\epsilon\pi}\sigma_\pi, \sigma_{\pi,\$}\right)' \tag{A.18}$$

$$\eta_{Y1,k\$} = 0_4 \tag{A.19}$$

$$\eta_{Y2,k\$} = -\frac{1}{\sigma_{\pi,\$}}\left(1, -\left(1-\frac{k}{\gamma}\right), -\frac{k}{\gamma}, 0\right)' \tag{A.20}$$

$$\alpha_{k\$} = \rho + \gamma\mu_\epsilon - \frac{1}{2}\gamma(\gamma+1)\sigma_\epsilon^2 - \gamma\rho_{\epsilon\pi}\sigma_\epsilon\sigma_\pi - \sigma_\pi^2 \tag{A.21}$$

$$\beta_{k\$} = \left(0, 1-\frac{k}{\gamma}, \frac{k}{\gamma}, \delta(1-\gamma)\right)' \tag{A.22}$$

$$\Psi_{k\$} = -\frac{1}{2}\frac{k}{\gamma}\left(\frac{k}{\gamma}-1\right)\frac{1}{\sigma_{\pi,\$}^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{A.23}$$

The matrix $\Psi_{k\$}$ is positive semidefinite because $k/\gamma \le 1$. Note that $\psi_{k\$}$ is singular and $\psi_{k\$} = 0_{4\times 4}$ if $k = 0$ or $k = \gamma$.

$\square$

Quadratic Gaussian Term Structure Models

Here we use the same notation as Ahn, Dittmar, and Gallant (2002).[13] Let $Y(t)$ denote a $N-$dimensional vector of state variables and $Z_M(t)$ a $M-$dimensional vector of independent Brownian motions.

Assumption 1. *The dynamics of the stochastic discount factor $SDF(t)$ are*[14]

$$\frac{dSDF(t)}{SDF(t)} = -r(t)\, dt + 1'_M\, diag\left[\eta_{0m} + \eta'_{Ym}Y(t)\right]_M\, dZ_M(t) \tag{A.24}$$

with

$$\eta_0 = (\eta_{01}, \ldots, \eta_{0M})' \in \mathcal{R}^M \tag{A.25}$$

$$\eta_Y = (\eta_{Y1}, \ldots, \eta_{YM})' \in \mathcal{R}^{M\times N} \tag{A.26}$$

Hence, the market price of risk is an affine function of the state vector $Y(t)$.

Assumption 2. *The short rate is a quadratic function of the state variables:*

$$r(t) = \alpha + \beta'Y(t) + Y(t)'\Psi Y(t), \tag{A.27}$$

where α is a constant, β is an N-dimensional vector of constants, and Ψ is an $N \times N$ dimensional positive semidefinite matrix of constants.[15]

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[13]In contrast to Ahn, Dittmar, and Gallant (2002): (i) we assume that the vector of Brownian motions driving the discount factor is identical to the vector of Brownian motions driving the state variables and thus Υ is the identify matrix, and (ii) we allow the vector of Brownian motions to have a dimension that is different from the number of state variables.

[14]An apostrophe denotes the transpose of a vector or matrix, $1'_M$ denotes a vector of ones, and $diag\left[Y'_m\right]_M$ denotes an M-dimensional matrix with diagonal elements $(Y_1, \ldots, Y_m)$.

[15]We don't impose an additional parameter restriction that guarantees non-negativity of the short rate.

If the matrix Ψ is non singular, then $r(t) \geq \alpha - \frac{1}{2}\beta'\Psi^{-1}\beta \; \forall t$.

Assumption 3. *The state vector $Y(t)$ follows a multidimensional OU-process:*

$$dY(t) = (\mu + \xi Y(t)) \; dt + \Sigma dZ_M(t), \tag{A.28}$$

where μ is an N-dimensional vector of constants, ξ is an N-dimensional square matrix of constants, and Σ is a $N \times M$-dimensional matrix of constants. We assume that ξ is diagonalizable and has negative real components of eigenvalues. Specifically, $\xi = U\Lambda U^{-1}$ in which U is the matrix of N eigenvectors and Λ is the diagonal matrix of eigenvalues.

Let $V(t,\tau)$ denote the price of a zero-coupon bond and $y(t,\tau)$ the corresponding yield. Specifically,

$$V(t,\tau) = \mathrm{E}_t \left[\frac{\mathrm{SDF}(t+\tau)}{\mathrm{SDF}(t)} \right] \tag{A.29}$$

$$y(t,\tau) = -\frac{1}{\tau} \ln \left(V(t,\tau) \right). \tag{A.30}$$

The bond price and corresponding yield are given in the next proposition.

Proposition 9 (Quadratic Gaussian Term Structure Model). *Let $\delta_0 = -\Sigma\Upsilon\eta_0 = -\Sigma\eta_0$ and $\delta_Y = -\Sigma\Upsilon\eta_Y = -\Sigma\eta_Y$. The bond price is an exponential quadratic function of the state vector*

$$V(t,\tau) = exp \left\{ A(\tau) + B(\tau)'Y(t) + Y(t)'C(\tau)Y(t) \right\}, \tag{A.31}$$

where $A(\tau)$, $B(\tau)$, and $C(\tau)$ satisfy the ordinary differential equations,

$$\frac{dC(\tau)}{d\tau} = 2C(\tau)\Sigma\Sigma'C(\tau) + \left(C(\tau)(\xi - \delta_Y) + (\xi - \delta_Y)'C(\tau) \right) - \Psi \tag{A.32}$$

$$\frac{dB(\tau)}{d\tau} = 2C(\tau)\Sigma\Sigma'B(\tau) + (\xi - \delta_Y)'B(\tau) + 2C(\tau)(\mu - \delta_0) - \beta \tag{A.33}$$

$$\frac{dA(\tau)}{d\tau} = trace\left[\Sigma\Sigma'C(\tau) \right] + \frac{1}{2}B(\tau)'\Sigma\Sigma'B(\tau) + B(\tau)'(\mu - \delta_0) - \alpha, \tag{A.34}$$

in which $A(0) = 0$, $B(0) = 0_N$, and $C(0) = 0_{N \times N}$. Moreover, the yield is a quadratic function of the state vector $Y(t)$:

$$y(t,\tau) = A_y(\tau) + B_y(\tau)'Y(t) + Y(t)'C_y(\tau)Y(t) \tag{A.35}$$

with $A_y(\tau) = -A(\tau)/\tau$, $B_y(\tau) = -B(\tau)/\tau$, and $C_y(\tau) = -C(\tau)/\tau$.

Proof. See Ahn et al. (2002). □

If the short rate is an affine function of the state vector $Y(t)$, then the bond price is an exponential affine function of the state vector $Y(t)$ because $\Psi = 0_{N \times N}$ implies $C(\tau) = 0_{N \times N}$ for all τ. The bond price in this case belongs to the class of essential affine term structure models (see Duffee (2002)) and is given in the next corollary.

Proposition 10 (Essential Affine Term Structure Model). *Let $\Psi = 0_{N \times N}$, $\delta_0 = -\Sigma\Upsilon\eta_0 = -\Sigma\eta_0$ and $\delta_Y = -\Sigma\Upsilon\eta_Y = -\Sigma\eta_Y$ and assume that $(\xi - \delta_Y)$ is invertible. The bond price is an exponential affine function of the state vector*

$$V(t,\tau) = exp \left\{ A(\tau) + B(\tau)'Y(t) \right\}, \tag{A.36}$$

where

$$B(\tau) = -\left((\xi - \delta_Y)'\right)^{-1}\left(e^{(\xi-\delta_Y)'\tau} - I_{N\times N}\right)\beta, \tag{A.37}$$

$I_{N\times N}$ *denotes the* N *dimensional identity matrix, and*

$$\begin{aligned}
A(\tau) = {} & \frac{1}{2}\beta'\left(\int_0^\tau \left(e^{(\xi-\delta_Y)'u}\right)' K\, e^{(\xi-\delta_Y)'u}\, du\right)\beta \\
& - \left(\beta'K + (\mu - \delta_0)'\left((\xi - \delta_Y)'\right)^{-1}\right)\left(\int_0^\tau e^{(\xi-\delta_Y)'u}\, du\right)\beta \\
& + \left(\frac{1}{2}\beta'K\,\beta + (\mu - \delta_0)'\left((\xi - \delta_Y)'\right)^{-1}\beta - \alpha\right)\tau
\end{aligned} \tag{A.38}$$

with

$$K = \left(\left((\xi - \delta_Y)'\right)^{-1}\right)'\Sigma\Sigma'\left((\xi - \delta_Y)'\right)^{-1}. \tag{A.39}$$

If $(\xi - \delta_Y)'$ *is diagonalizable; i.e.* $(\xi - \delta_Y)' = T\Lambda T^{-1}$ *then*[16]

$$B(\tau) = -T\, diag\left[\frac{1}{\lambda_i}\left(e^{\lambda_i\tau} - 1\right)\right]T^{-1}\beta, \tag{A.40}$$

$$\int_0^\tau e^{(\xi-\delta_Y)'u}\, du = T\, diag\left[\frac{1}{\lambda_i}\left(e^{\lambda_i\tau} - 1\right)\right]T^{-1}, \tag{A.41}$$

and

$$\int_0^\tau \left(e^{(\xi-\delta_Y)'u}\right)' K\, e^{(\xi-\delta_Y)'u}\, du = \left(T^{-1}\right)' G(\Lambda, t)T^{-1}, \tag{A.42}$$

where $G(\Lambda, t)$ *is a* $m \times m$-*matrix with elements given by*

$$G_{ij} = \frac{\omega_{ij}}{\lambda_i + \lambda_j}\left(e^{(\lambda_i+\lambda_j)t} - 1\right) \tag{A.43}$$

and ω_{ij} *denotes the element of the matrix* $\Omega = T'KT$ *in the* i^{th}-*row and* j^{th}-*column.*
    Moreover, the yield is an affine function of the state vector $Y(t)$:

$$y(t, \tau) = A_y(\tau) + B_y(\tau)'Y(t) \tag{A.44}$$

with $A_y(\tau) = -A(\tau)/\tau$, *and* $B_y(\tau) = -B(\tau)/\tau$.

Proof. *where* $A(\tau)$ *and* $B(\tau)$ *satisfy the ordinary differential equations,*

$$\frac{dB(\tau)}{d\tau} = (\xi - \delta_Y)'B(\tau) - \beta \tag{A.45}$$

$$\frac{dA(\tau)}{d\tau} = \frac{1}{2}B(\tau)'\Sigma\Sigma'B(\tau) + B(\tau)'(\mu - \delta_0) - \alpha, \tag{A.46}$$

in which $A(0) = 0$ *and* $B(0) = 0_N$. $\square$

---

[16]The matrix $(\xi - \delta_Y)$ is invertible and thus all eigenvalues are nonzero.

Table 1: **Parameter Choice for Expected Inflation Disagreement Example.**

Parameter	Description	Value
Investors		
ρ	Time Preference Parameter	0.017
γ	Common Risk Aversion	7
δ	Habit Parameter	0.070
$f(0)$	Initial Consumption Allocation	0.5
Consumption		
μ_ϵ	Expected Consumption Growth	0.0172
σ_ϵ	Volatility of Consumption	0.0332
Inflation		
σ_π	Inflation Volatility	0.013
$\bar{x}$	Long Run Mean of Expected Inflation	0.030
κ	Mean Reversion of Expected Inflation	0.35
σ_x	Volatility of Expected Inflation	0.014
$\rho_{\epsilon\pi}$	ρ of Real Consumption Growth and Realized Inflation	-0.2
$\rho_{x\pi}$	ρ of Expected Inflation and Realized Inflation	0
$\rho_{x\epsilon}$	ρ of Expected Inflation and Real Consumption Growth	0
Disagreement		
$\bar{x}^1$	Long run mean of first investor	$\bar{x} - \frac{1}{2}\Delta_{\bar{x}}$
$\bar{x}^2$	Long run mean of second investor	$\bar{x} + \frac{1}{2}\Delta_{\bar{x}}$
κ^1	Mean reversion of first investor	$\kappa - \frac{1}{2}\Delta_\kappa$
κ^2	Mean reversion of first investor	$\kappa + \frac{1}{2}\Delta_\kappa$
$\Delta_{\bar{x}}$	$\bar{x}^2 - \bar{x}^1$	0.01
Δ_κ	$\kappa^2 - \kappa^1$	-0.3

Table 2: **Inflation Beliefs Dispersion and Real Yields I.** The table reports results from OLS regressions of the determinants of real yields and volatilities of real yields. Real yields, are from Chernov and Mueller (2012) and Gürkaynak et al. (2010). Real yield volatilities are estimated from a GARCH(1,1). Explanatory variables include inflation belief dispersion (Dispersion) and the mean of the inflation forecast (Mean Inflation). The t-statistics are Newey-West corrected. The mean and dispersion of monthly inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: Q1 1978 - Q4 2011 (Panel A) and Q2 1981 - Q4 2011 (Panel B).

Panel A: Full Sample (Q1 1978 to Q4 2011)

Real Yield Level Regressions

	1 Year		2 Year		3 Year		5 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	0.018	0.017	0.003	0.003	0.006	0.006	0.011	0.012
t-statistic	1.926	1.844	0.324	0.452	0.689	0.900	1.670	2.223
Dispersion	0.141	0.316	0.334	0.602	0.301	0.573	0.234	0.509
t-statistic	0.841	1.216	2.130	3.091	2.052	3.263	1.825	3.470
Mean Inflation		-0.202		-0.357		-0.362		-0.367
t-statistic		-0.984		-2.427		-2.635		-2.974
Adj. R^2	0.014	0.024	0.120	0.177	0.118	0.190	0.100	0.209
N	100	100	136	136	136	136	136	136

Real Yield Volatility Regressions

	1 Year		2 Year		3 Year		5 Year	
Intercept	0.000	0.000	0.002	0.003	0.002	0.002	0.002	0.003
t-statistic	-0.084	-0.147	1.647	1.659	2.128	2.149	3.296	3.376
Dispersion	0.150	0.165	0.095	0.114	0.075	0.089	0.050	0.062
t-statistic	3.198	2.992	2.991	3.331	3.119	3.591	3.084	3.602
Mean Inflation		-0.017		-0.026		-0.020		-0.016
t-statistic		-0.257		-0.560		-0.607		-0.806
Adj. R^2	0.305	0.300	0.204	0.205	0.247	0.249	0.267	0.275
N	100	100	136	136	136	136	136	136

Panel B: No High Inflation Period Sample (Q2 1981 to Q4 2011)

Real Yield Level Regressions

	1 Year		2 Year		3 Year		5 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	0.003	-0.013	-0.011	-0.015	-0.008	-0.011	-0.001	-0.004
t-statistic	0.315	-1.453	-1.199	-1.560	-0.989	-1.400	-0.223	-0.607
Dispersion	0.443	-0.089	0.625	0.551	0.587	0.520	0.504	0.460
t-statistic	2.338	-0.440	4.019	2.890	4.287	3.118	4.700	3.404
Mean Inflation		1.177		0.203		0.184		0.120
t-statistic		4.224		0.770		0.830		0.671
Adj. R^2	0.156	0.330	0.287	0.289	0.309	0.310	0.329	0.328
N	87	87	123	123	123	123	123	123

Real Yield Volatility Regressions

	1 Year		2 Year		3 Year		5 Year	
Intercept	0.000	-0.002	0.002	0.003	0.002	0.002	0.002	0.002
t-statistic	-0.197	-0.592	1.925	1.361	2.341	1.645	3.192	2.303
Dispersion	0.154	0.117	0.095	0.104	0.077	0.080	0.054	0.055
t-statistic	3.553	2.481	3.418	2.982	3.558	3.337	3.459	3.482
Mean Inflation		0.082		-0.024		-0.010		-0.003
t-statistic		0.950		-0.354		-0.238		-0.100
Adj. R^2	0.410	0.422	0.213	0.210	0.280	0.275	0.324	0.319
N	87	87	123	123	123	123	123	123

Table 3: **Inflation Beliefs Dispersion and Real Yields II.** The table reports results from OLS regressions of the determinants of real yields and volatilities of real yields. Real yields, are computed from the CRSP Risk-Free Rates File (1- and 3-months nominal yields based on bid/ask average prices), the Fama-Bliss Discount Bond File (nominal yields with 1 to 5 years to maturity based on artificial discount bonds) and realized inflation (based on the CPI). Real yield volatilities are estimated from a GARCH(1,1). Explanatory variables include inflation belief dispersion (Dispersion) and the mean of the inflation forecast (Mean Inflation). The t-statistics are Newey-West corrected. The mean and dispersion of monthly inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: Jan 1978 - Dec 2011 (Panel A) and Apr 1981 - Dec 2011 (Panel B).

Panel A: Full Sample (January 1978 to December 2011)

Real Yield Level Regressions

	1 Year		2 Year		3 Year		4 Year		5 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	-0.004	-0.003	-0.001	0.001	0.003	0.006	0.007	0.009	0.009	0.012
t-statistic	-0.416	-0.306	-0.054	0.154	0.324	0.684	0.637	1.161	0.874	1.560
Dispersion	0.455	0.957	0.438	1.060	0.404	1.129	0.377	1.164	0.356	1.184
t-statistic	2.242	3.934	2.076	4.322	1.853	4.636	1.707	4.822	1.617	5.036
Mean Inflation		-0.679		-0.842		-0.981		-1.065		-1.120
t-statistics		-2.411		-2.931		-3.401		-3.687		-3.928
Adj. R^2	0.084	0.167	0.075	0.200	0.063	0.232	0.055	0.255	0.050	0.278
N	400	400	400	400	400	400	400	400	400	400

Real Yield Volatility Regressions

	1 Year		2 Year		3 Year		4 Year		5 Year	
Intercept	0.000	0.000	0.002	0.002	0.002	0.002	0.003	0.003	0.003	0.003
t-statistic	0.340	0.216	4.029	3.814	6.274	6.164	7.878	7.617	8.802	8.530
Dispersion	0.103	0.065	0.068	0.042	0.058	0.031	0.053	0.031	0.049	0.027
t-statistic	5.423	2.671	6.938	2.667	7.736	2.483	8.115	2.460	8.336	2.323
Mean Inflation		0.051		0.035		0.036		0.029		0.029
t-statistic		1.763		2.064		2.693		2.273		2.394
Adj. R^2	0.338	0.374	0.361	0.401	0.369	0.432	0.342	0.388	0.327	0.378
N	400	400	400	400	400	400	400	400	400	400

Panel B: No High Inflation Period Sample (April 1981 to December 2011)

Real Yield Level Regressions

	1 Year		2 Year		3 Year		4 Year		5 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	-0.017	-0.037	-0.016	-0.034	-0.014	-0.029	-0.012	-0.026	-0.010	-0.022
t-statistic	-1.784	-3.570	-1.826	-3.398	-1.736	-3.098	-1.592	-2.851	-1.402	-2.593
Dispersion	1.274	0.865	1.323	0.966	1.336	1.031	1.338	1.064	1.323	1.080
t-statistic	7.153	3.936	7.887	4.436	8.501	4.882	9.055	5.197	9.576	5.510
Mean Inflation		1.066		0.930		0.796		0.714		0.633
t-statistic		2.787		2.454		2.161		1.984		1.837
Adj. R^2	0.407	0.475	0.439	0.490	0.467	0.505	0.489	0.522	0.509	0.536
N	369	369	369	369	369	369	369	369	369	369

Real Yield Volatility Regressions

	1 Year		2 Year		3 Year		4 Year		5 Year	
Intercept	0.001	0.000	0.002	0.002	0.003	0.002	0.003	0.002	0.003	0.002
t-statistic	1.666	0.204	4.381	2.218	2.198	0.681	7.200	4.423	8.111	5.125
Dispersion	0.068	0.046	0.046	0.031	0.052	0.020	0.037	0.026	0.035	0.024
t-statistic	3.924	3.830	3.818	3.696	1.585	1.273	4.262	3.912	4.302	3.965
Mean Inflation		0.058		0.040		0.083		0.030		0.028
t-statistic		2.402		2.394		1.218		2.597		2.591
Adj. R^2	0.338	0.396	0.328	0.386	0.041	0.065	0.356	0.412	0.360	0.414
N	369	369	369	369	369	369	369	369	369	369

Table 4: **Inflation Beliefs Dispersion and Nominal Yields.** The table reports results from OLS regressions of the determinants of nominal yields and volatilities of nominal yields. Nominal yields are from the CRSP Risk-Free Rates File (1- and 3-months nominal yields based on bid/ask average prices) and the Fama-Bliss Discount Bond File (nominal yields with 1 to 5 years to maturity based on artificial discount bonds). Nominal yield volatilities are estimated from a GARCH(1,1). Explanatory variables include inflation belief dispersion (Dispersion) and the mean of the inflation forecast (Mean Inflation). The t-statistics are Newey-West corrected. The mean and dispersion of inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: Jan 1978 - Dec 2011 (Panel A) and Apr 1981 - Dec 2011 (Panel B).

Panel A: Full Sample (January 1978 to December 2011)

Nominal Yield Level Regressions

	1 Year		2 Year		3 Year		4 Year		5 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	-0.016	-0.017	-0.012	-0.013	-0.008	-0.009	-0.005	-0.005	-0.002	-0.003
t-statistic	-2.088	-2.228	-1.710	-1.790	-1.219	-1.267	-0.759	-0.793	-0.387	-0.411
Dispersion	1.312	0.944	1.295	1.047	1.262	1.115	1.235	1.148	1.212	1.163
t-statistic	10.108	4.456	10.482	4.963	10.550	5.411	10.635	5.713	10.938	6.041
Mean Inflation		0.502		0.339		0.201		0.118		0.067
t-statistic		2.485		1.648		0.983		0.583		0.338
Adj. R^2	0.495	0.526	0.506	0.520	0.514	0.518	0.522	0.523	0.536	0.536
N	408	408	408	408	408	408	408	408	408	408

Nominal Yield Volatility Regressions

	1 Year		2 Year		3 Year		4 Year		5 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	-0.003	-0.003	-0.001	-0.001	0.000	0.000	0.000	0.001	0.001	0.001
t-statistic	-2.660	-2.611	-0.934	-0.922	0.089	0.050	0.903	0.906	1.518	1.530
Dispersion	0.125	0.111	0.084	0.080	0.069	0.063	0.063	0.067	0.055	0.062
t-statistic	5.464	5.426	5.293	5.308	5.039	5.509	5.480	5.374	5.404	5.575
Mean Inflation		0.019		0.006		0.009		-0.006		-0.010
t-statistic		0.704		0.321		0.511		-0.396		-0.698
Adj. R^2	0.454	0.457	0.432	0.432	0.419	0.420	0.412	0.412	0.389	0.392
N	408	408	408	408	408	408	408	408	408	408

Panel B: No High Inflation Period Sample (April 1981 to December 2011)

Nominal Yield Level Regressions

	1 Year		2 Year		3 Year		4 Year		5 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	-0.016	-0.039	-0.015	-0.036	-0.013	-0.032	-0.011	-0.028	-0.009	-0.024
t-statistic	-1.631	-3.761	-1.665	-3.597	-1.570	-3.310	-1.419	-3.074	-1.224	-2.833
Dispersion	1.303	0.830	1.351	0.931	1.365	0.996	1.367	1.029	1.352	1.045
t-statistic	7.044	3.795	7.757	4.291	8.362	4.732	8.890	5.038	9.394	5.345
Mean Inflation		1.231		1.095		0.961		0.879		0.798
t-statistic		3.257		2.922		2.638		2.467		2.341
Adj. R^2	0.407	0.493	0.438	0.506	0.466	0.520	0.488	0.535	0.508	0.549
N	369	369	369	369	369	369	369	369	369	369

Nominal Yield Volatility Regressions

	1 Year		2 Year		3 Year		4 Year		5 Year	
	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
Intercept	-0.002	-0.003	0.000	-0.001	0.001	0.000	0.001	0.000	0.001	0.001
t-statistic	-2.110	-2.602	-0.155	-0.855	1.376	0.326	1.183	0.442	1.700	0.966
Dispersion	0.104	0.083	0.073	0.061	0.057	0.047	0.060	0.053	0.055	0.049
t-statistic	5.385	5.179	5.143	4.960	5.314	5.275	5.209	4.948	5.467	5.176
Mean Inflation		0.055		0.031		0.026		0.019		0.014
t-statistic		2.283		1.917		1.817		1.476		1.411
Adj. R^2	0.441	0.470	0.424	0.442	0.410	0.428	0.410	0.418	0.404	0.409
N	369	369	369	369	369	369	369	369	369	369

Figure 1: **Expected Inflation Disagreement Example - Average Yield Levels and Average Yield Volatilities.** The figure shows average yield levels (top plots) and average yield volatilities (bottom plots) for nominal (left plots) and real (right plots) yield curves as a function of maturity in years. Common beliefs and disagreement cases of the model are plotted in addition to the data. "Common Beliefs - Investor 1" ("Common Beliefs - Investor 2") plots yield curve properties under an economy entirely populated by investor 1 (2). "Disagreement" plots yield curve properties in an economy populated by both agents. The model averages are computed by simulating 2000 paths of the economy. The data is computed from the Fama-Bliss Discount Bond File from April 1981 to December 2011. Parameters used are found in Table 1 and in Section 4.1.

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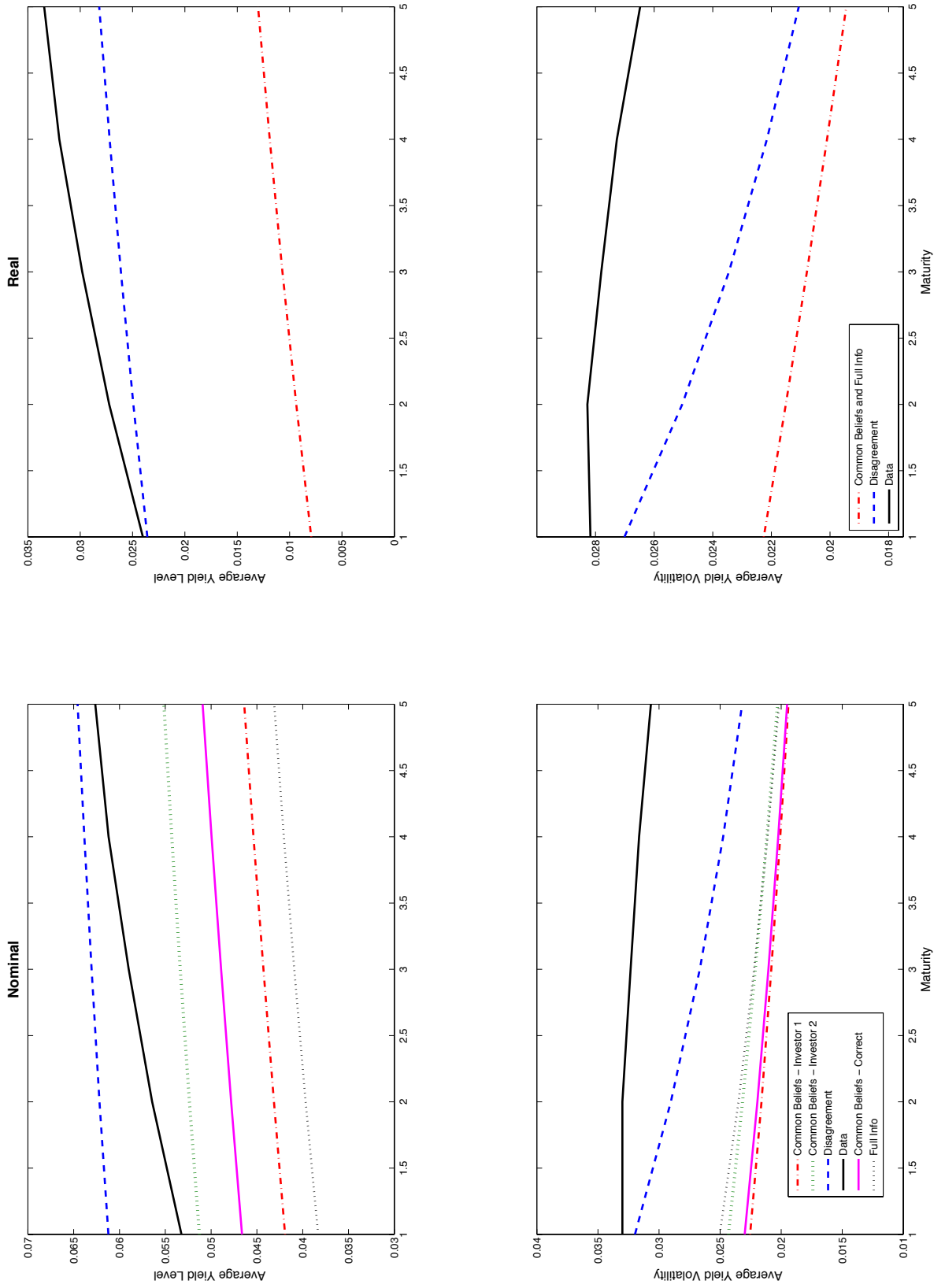
Figure 2: **Expected Inflation Disagreement Example - Yield Levels and Yield Volatilities with Varying Disagreement.** The figure shows yield levels (top plots) and yield volatilities (bottom plots) for nominal (left plots) and real (right plots) yield curves as a function of maturity in years. Yield curve properties are plotted for three different levels of $t = 0$ expected inflation disagreement $\Delta(0) = \frac{x^2(0) - x^1(0)}{\sigma_{\pi,\$}}$: "Zero Disagreement" $\left(\Delta(0) = \frac{0.03 - 0.03}{\sigma_{\pi,\$}} \right)$, "Steady-State Disagreement" $\left(\Delta(0) = \frac{0.0309 - 0.0279}{\sigma_{\pi,\$}} \right)$, and "High Disagreement" $\left(\Delta(0) = \frac{0.0336 - 0.0253}{\sigma_{\pi,\$}} \right)$. The other parameters used are found in Table 1 and in Section 4.1.
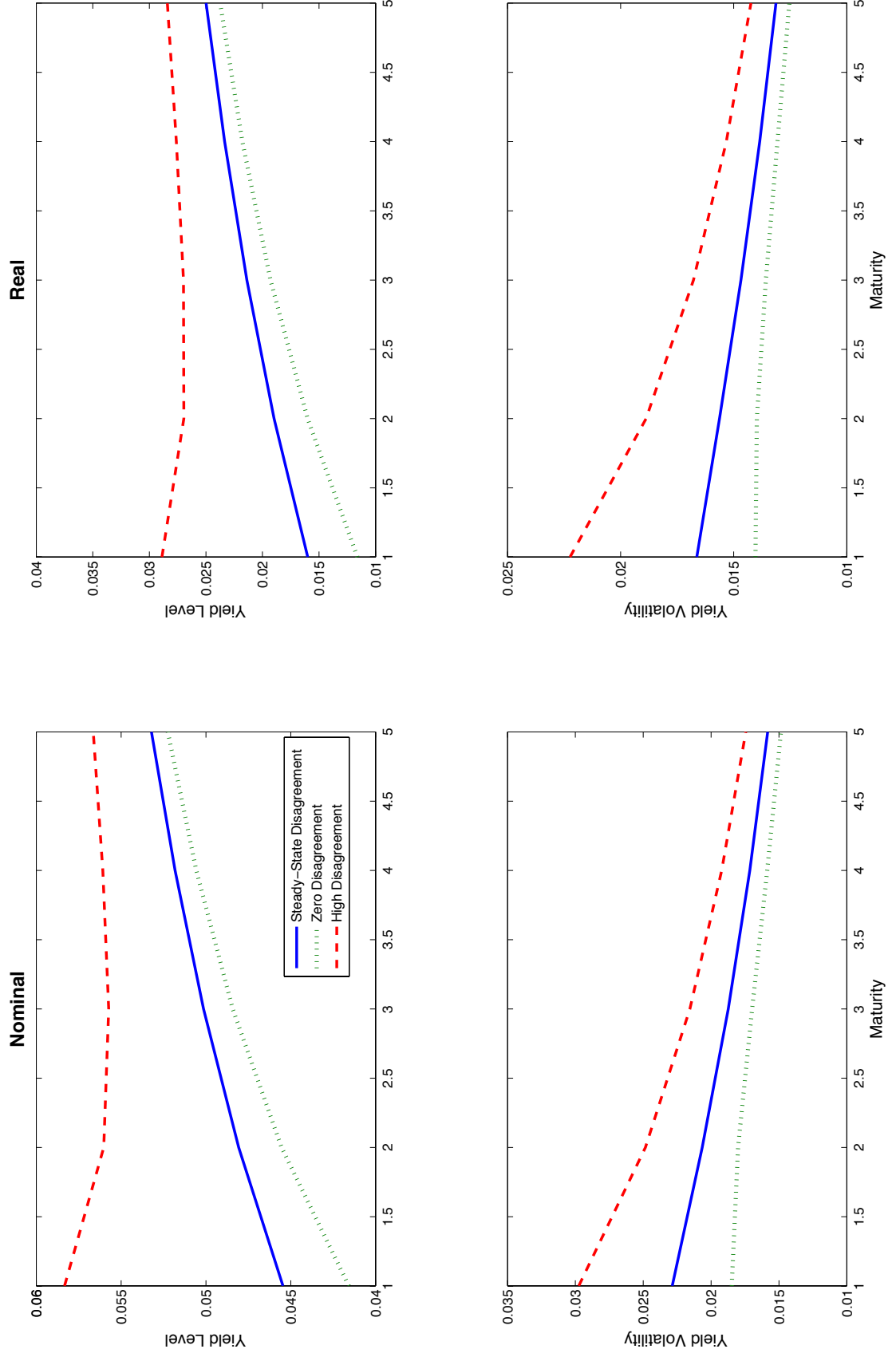
Figure 3: **Expected Inflation Disagreement Example – Nominal and Real Yield Levels versus the Sharing Rule.** The figure shows nominal and real yield levels for various maturities as function of the sharing rule $f(0)$. Parameters used are found in Table 1 and in Section 4.1.
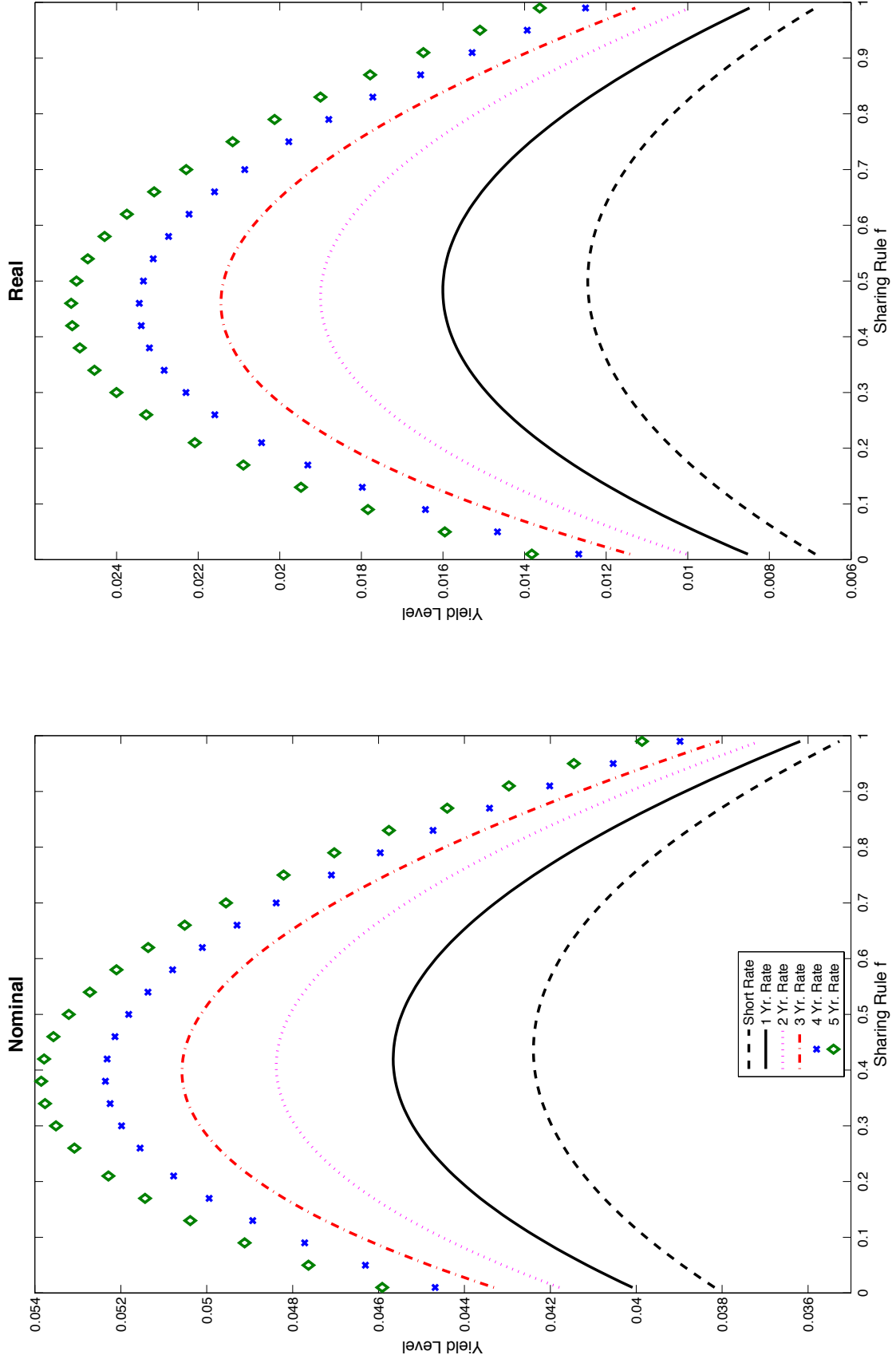
Figure 4: **Expected Inflation Disagreement Example - Impact of Inflation Expectations on Real Yields.** The figure shows four plots of real yield levels for various maturities as function of expected inflation x^1 as perceived by the first investor. Each plot captures different beliefs about the expected inflation model through $\Delta_{\bar{x}} = \bar{x}^2 - \bar{x}^1$ and $\Delta_{\kappa} = \kappa^2 - \kappa^1$. Parameters used are found in Table 1 and in Section 4.1.
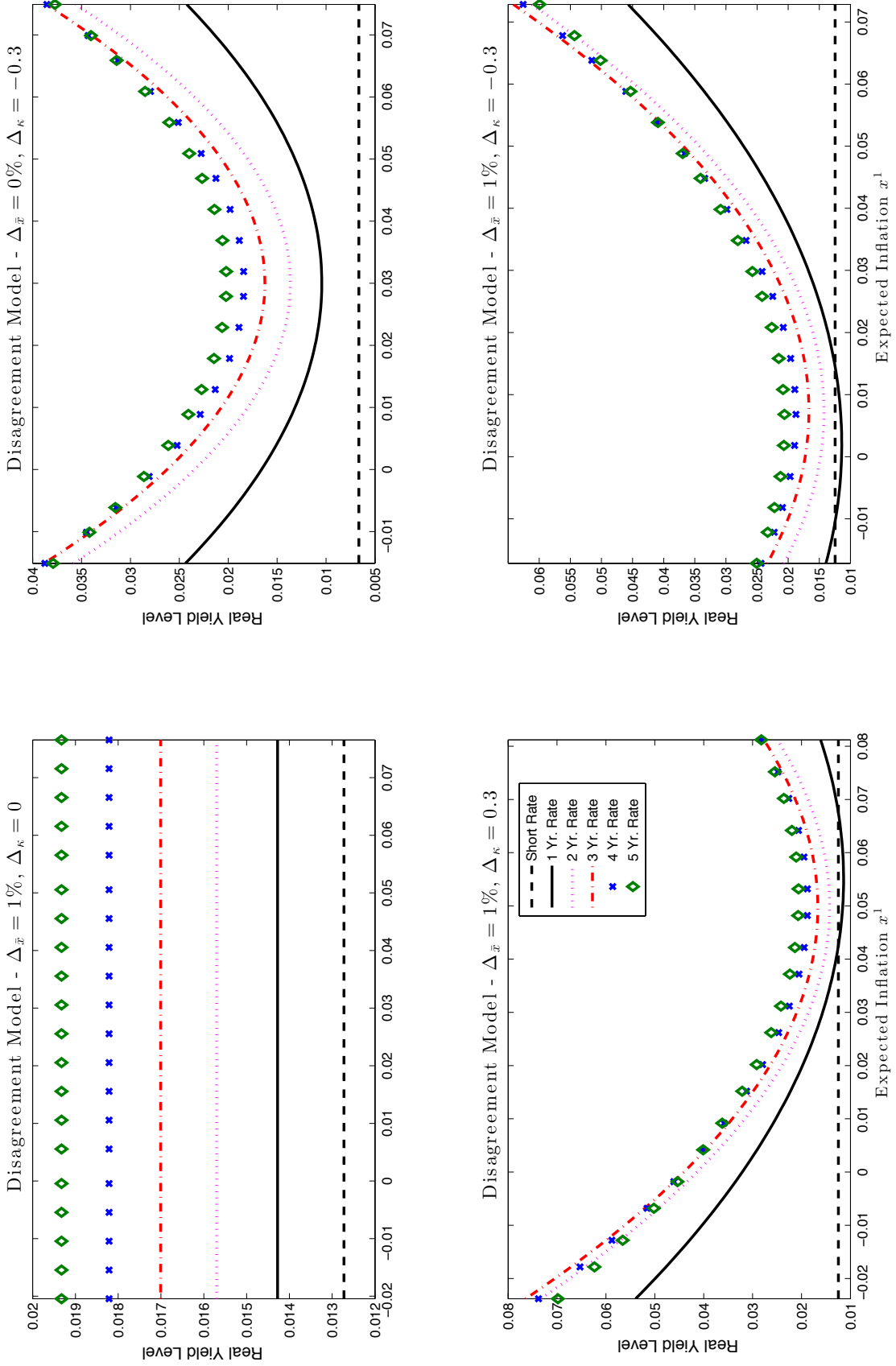
Figure 5: **Mean Inflation Beliefs and Inflation Beliefs Dispersion.** The figure shows mean inflation forecasts and inflation based on the CPI (top plot) and inflation forecast dispersions (bottom plot), the standard deviation of forecasts around the mean forecast —based on Michigan Surveys of Consumers— with NBER recessions as gray shaded areas. Monthly inflation is plotted one year ahead (until July 2012). The mean and dispersion of monthly inflation forecasts are computed from raw data obtained from Thomson Reuters / University of Michigan. Sample: January 1978 - December 2011.
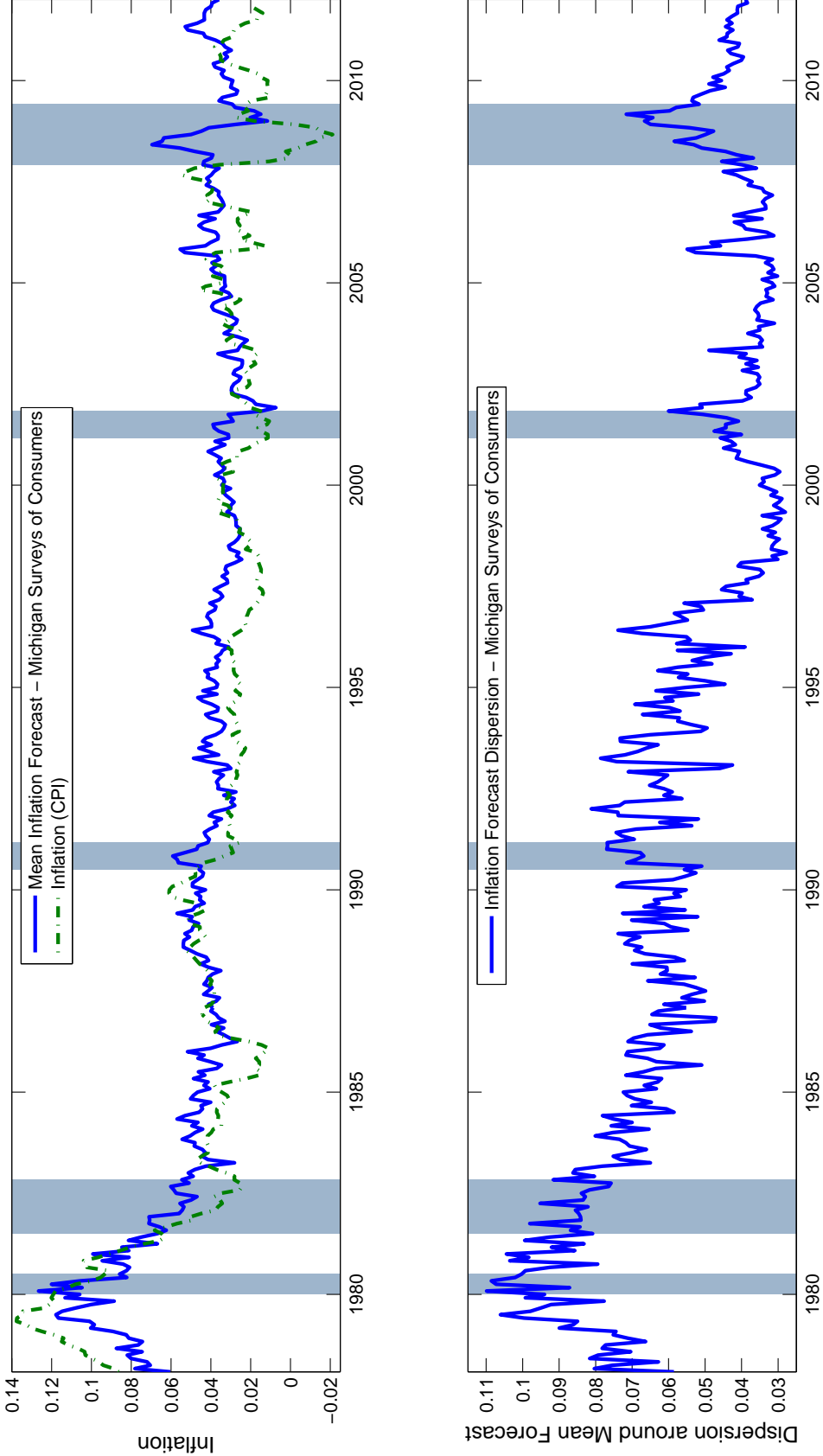
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