Timing Decisions in Organizations: Communication and Authority in a Dynamic Environment*

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Abstract

We consider a problem where an uninformed principal makes a timing decision interacting with an informed but biased agent. Because time is irreversible, the direction of the bias crucially affects the agent's ability to credibly communicate information. When the agent favors late decision-making, full information revelation often occurs. In this case, centralized decision-making, where the principal retains authority and communicates with the agent, implements the optimal full-commitment solution, making delegation weakly suboptimal. When the agent favors early decision-making, communication is partial, while decisions are unbiased or delayed. Delegation is optimal if the bias is small or delegation can be timed.

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1 Introduction

Many decisions in organizations deal with the optimal timing of taking a certain action. Because information in organizations is dispersed, the decision-maker needs to rely on the information of her better-informed subordinates who, however, may have conflicting preferences. Consider the following two examples of such settings. 1) In a typical hierarchical firm, top executives may be less informed than the product manager about the optimal timing of the launch of a new product. It would not be surprising for an empire-building product manager to be biased in favor of an earlier launch. 2) The CEO of a multinational corporation is contemplating when to shut down a plant in a struggling economic region. While the local plant manager is better informed about the prospects of the plant, he may be biased towards a later shutdown due to personal costs of relocation.

These examples share a common theme. An uninformed principal faces an optimal stopping-time problem – when to exercise a real option. An agent is better informed than the principal but is biased towards earlier or later exercise. In this paper, we study how organizations make timing decisions in such a setting. First, we examine the effectiveness of centralized decision-making, where the principal retains authority and gets information by repeatedly communicating with the agent. Next, we compare this with decentralized decision-making, where the principal delegates the decision to the agent, and develop implications for the optimal allocation of authority.

We show that the economics underlying this problem are quite different from those when the decision is static rather than dynamic, and the decision variable is scale of the action rather than a stopping time, which has been the focus of most of the existing literature. For stopping time decisions, the key determinant of the optimal allocation of authority is the direction of the agent's bias: If the agent favors late exercise, such as in the case of a plant closure, centralized decision-making is very efficient and always dominates delegation within our framework. In contrast, if the agent favors early exercise, such as in the case of a product launch, delegation has benefits and frequently dominates centralized decision-making.

Our setting combines the framework of real option exercise problems with the framework of cheap talk communication between the principal and the agent (Crawford and Sobel, 1982). The principal must decide when to exercise an option whose payoff depends on an unknown parameter. The agent knows the parameter, but the agent's payoff from exercise differs from the principal's due to a bias. As a benchmark, we start by analyzing the optimal mechanism if the principal could commit to any decision rule but cannot make monetary transfers to the agent, and show that it takes the form of "interval delegation," similar to static problems.[1] If the agent favors later exercise than the principal, the optimal mechanism features the agent's most desired timing if it is early enough, and pools all types whose most desired timing is too late. If the agent favors earlier exercise than the principal, the optimal mechanism is the opposite: it features the agent's most desired timing if it is late enough, and pools all types whose most desired timing is too early. We next examine under what conditions the principal is able to implement this optimal decision rule even if he lacks full commitment power.

We first consider centralized decision-making, where the principal has no commitment power at all and only relies on informal "cheap talk" communication with the agent while retaining authority over the decision. At any point in time, the agent sends a message to the principal about whether or not to exercise the option. Conditional on the received message and the history of the game, the principal chooses whether to exercise or wait. In equilibrium, the agent's communication strategy and the principal's exercise decisions are mutually optimal, and the principal rationally updates her beliefs about the agent's private information.

Our main result is that centralized decision-making implements the full-commitment optimal mechanism if the agent is biased towards late exercise, but not if the agent is biased towards early exercise. The intuition for this result lies in the nature of time as a decision variable: While the principal always has the choice to exercise at a point later than the present, she cannot do the reverse, i.e., exercise at a point earlier than the present. If the agent is biased towards late exercise, he can withhold information and reveal it later, exactly at the point where he finds it optimal to exercise the option. When the agent with a late exercise bias recommends exercise, the principal learns that it is too late to do so and is tempted to go back in time and exercise the option in the past. This, however, is not feasible, and hence the principal finds it optimal to follow the agent's recommendation. Knowing that, the agent communicates honestly, although communication occurs with delay. Thus, the inability to go back in time commits the principal to follow the agent's recommendation, which leads to effective communication and allows the principal to implement the full-commitment optimal mechanism despite having no commitment power.

In contrast, if the agent is biased towards early exercise and recommends exercise at his most preferred time, the principal is tempted to delay the decision. Unlike changing past actions, changing future actions is possible, and hence time does not allow the principal to commit to follow

[1] E.g., Melumad and Shibano (1991), Alonso and Matouschek (2008), and Amador and Bagwell (2013). See the end of this section for the discussion of the related literature.

the agent's recommendation. Expecting that the principal would not follow his recommendation if he recommends to exercise the option at his most desired time, the agent deviates from that strategy. Therefore, only partial information revelation is possible and the optimal mechanism cannot be implemented if the principal has no commitment power.

The asymmetric nature of time has important implications for the optimal allocation of authority in organizations. In particular, we examine the principal's choice between delegating decision-making rights to the agent and retaining authority and communicating with the agent – the problem studied by Dessein (2002) in the context of static decisions. Because centralized decision-making implements the optimal mechanism when the agent favors late exercise, the principal is always better off keeping authority and communicating with the agent, rather than delegating the decision to the agent in this case. This result is different from the result for static decisions, where delegation is usually optimal if the agent's bias is sufficiently small (Dessein, 2002). Conversely, if the agent favors early exercise, as in the case of a product launch, delegation may dominate centralized decision-making. Intuitively, because in this case time does not have valuable commitment power, communication is not as efficient as in the case of a late exercise bias. As a consequence, delegation can now be optimal because it allows for more effective use of the agent's information. We show that the trade-off between information and bias makes delegation superior when the agent's bias is sufficiently small, similar to the result for static decisions.

We next allow the principal to time the delegation decision strategically, i.e., to choose the optimal timing of delegating authority to the agent. When the agent favors late exercise, the principal finds it optimal to retain authority forever: her commitment power due to the inability to go back in time makes communication effective and eliminates the need for delegation. In contrast, when the agent favors early exercise, the principal finds it optimal to delegate authority to the agent at some point in time, and delegation occurs later when the agent's bias is higher. In fact, delegating authority at the right time implements the optimal mechanism, i.e., it is the optimal decision making rule in this case. This result further emphasizes that the direction of the agent's bias is the main driver of the allocation of authority for timing decisions. This is different from static decisions, such as choosing the scale of the project, where the magnitude of the agent's bias and the importance of his private information are usually the factors emphasized as the drivers of the optimal allocation of authority.

Finally, our paper provides implications about the equilibrium properties of communication and decision-making under centralization. When the agent favors late exercise, there is often full information revelation but delay in option exercise. Conversely, when the agent favors early exercise, there is partial revelation of information, while exercise is either unbiased or delayed. The comparative statics analysis for this case shows that an increase in volatility or in the growth rate of the option payoff, as well as a decrease in the discount rate, lead to less information being revealed in equilibrium. Intuitively, these changes increase the value of the option to delay exercise and thereby effectively increase the conflict of interest between the principal and the agent with an early exercise bias.

The paper proceeds as follows. The remainder of this section discusses the related literature. Section 2 describes the setup. Section 3 solves for the optimal mechanism if the principal can commit to decision rules, which is the benchmark for the rest of the analysis. Section 4 analyzes dynamic cheap talk communication and examines when it implements the optimal commitment mechanism. Section 5 examines delegation. Section 6 extends the model in two directions: allowing for a non-uniform distribution of types and allowing for public news about the agent's type. Finally, Section 7 concludes. The Appendix presents the proofs of most propositions and also gives a very simple example, analogous to the quadratic-uniform example in Crawford and Sobel (1982), which illustrates the main intuition and findings of the paper. The Online Appendix contains additional proofs and the analysis of alternative versions of the model.

Related literature

Our paper is related to the literature that analyzes decision-making in the presence of an informed but biased expert. The seminal paper in this literature is Crawford and Sobel (1982), who consider a cheap talk setting, where the expert sends a message to the decision-maker and the decision-maker cannot commit to the way she reacts to the message. Our paper differs from Crawford and Sobel (1982) in that communication between the expert and the decision-maker is dynamic and concerns the timing of option exercise, rather than a static decision such as choosing the scale of a project. To our knowledge, ours is the first paper that studies the problem of optimal timing in a cheap talk setting. Surprisingly, even though there is no flow of additional private information to the agent, equilibria differ substantially from those in Crawford and Sobel (1982). We benchmark our setting against the static communication problem in Section 4.3.1.

By studying the choice between communication and delegation, our paper contributes to the literature on authority in organizations (e.g., Holmstrom, 1984; Aghion and Tirole, 1997; Dessein, 2002; Alonso and Matouschek, 2008). Gibbons, Matouschek, and Roberts (2013), Bolton and Dewatripont (2013), and Garicano and Rayo (2015) provide comprehensive reviews of this literature. Unlike Crawford and Sobel (1982), where the principal has no commitment power, the papers in this literature allow the principal to have some degree of commitment, although most of them rule out contingent transfers to the agent. Our paper is most closely related to Dessein (2002), who assumes that the principal can commit to delegate full decision-making authority to the agent. Dessein (2002) studies the principal's choice between delegating the decision and communicating with the agent via cheap talk to make the decision himself, and shows that delegation dominates communication if the agent's bias is not too large. Relatedly, Harris and Raviv (2005, 2008) and Chakraborty and Yilmaz (2013) analyze the optimality of delegation in settings with two-sided private information. Alonso, Dessein, and Matouschek (2008, 2014) and Rantakari (2008) compare centralized and decentralized decision-making in a multidivisional organization that faces a trade-off between adapting divisions' decisions to local conditions and coordinating decisions across divisions.[2] Our paper contributes to this literature by studying delegation of timing decisions and showing that the optimality of delegation crucially depends on the direction of the agent's bias. In particular, unlike in the static problem, it is never optimal to delegate decisions where the agent has a delay bias. In contrast, delegating the decision at the right time implements the second-best if the agent has an early exercise bias.

Other papers in this literature assume that the principal can commit to a decision rule and thus focus on a partial form of delegation: the principal offers the agent a set of decisions from which the agent can choose her preferred one. These papers include Holmstrom (1984), Melumad and Shibano (1991), Alonso and Matouschek (2008), Goltsman et al. (2009), Amador and Bagwell (2013), and Frankel (2014). In Baker, Gibbons, and Murphy (1999) and Alonso and Matouschek (2007), the principal's commitment power arises endogenously through relational contracts. Guo (2015) studies the optimal mechanism without transfers in an experimentation setting where the agent prefers to experiment longer than the principal. The optimal contract in her paper is time-consistent but becomes time-inconsistent if the agent prefers to experiment less than the principal, which is related to the asymmetry of our results in the direction of the agent's bias.[3] Our paper differs from this literature because it focuses on the situation where the principal has no or little commitment and communicates with the agent.

[2] See also Dessein, Garicano, and Gertner (2010) and Friebel and Raith (2010). Dessein and Santos (2006) study the benefits of specialization in the context of a similar trade-off, but do not analyze strategic communication.

[3] Halac, Kartik, and Liu (2015) also analyze optimal dynamic contracts in an experimentation problem, but in a different setting and allowing for transfers.

Several papers analyze dynamic extensions of Crawford and Sobel (1982). In Sobel (1985), Benabou and Laroque (1992), and Morris (2001), the advisor's preferences are unknown and his messages in prior periods affect his reputation with the decision-maker.[4] Aumann and Hart (2003), Krishna and Morgan (2004), Goltsman et al. (2009), and Golosov et al. (2014) consider settings with persistent private information where the principal actively participates in communication by either sending messages himself or taking an action following each message of the advisor.[5] Our paper differs from this literature because of the dynamic nature of the decision problem: the decision variable is the timing of option exercise, rather than a static variable. The inability to go back in time creates an implicit commitment device for the principal to follow the advisor's recommendations and thereby improves communication, a feature not present in prior literature.

Finally, our paper is related to the literature on option exercise in the presence of agency problems. Grenadier and Wang (2005), Gryglewicz and Hartman-Glaser (2015), and Kruse and Strack (2015) study such settings but assume that the principal can commit to contracts and make contingent transfers to the agent, which makes the problem conceptually different from ours. Several papers study signaling through option exercise.[6] They assume that the decision-maker is informed, while in our setting the decision-maker is uninformed, unless the principal delegates the decision to the agent.

2 Model setup

A firm (or an organization, more generally) has a project and needs to decide on the optimal time to implement it. There are two players, the uninformed party (principal, P) and the informed party (agent, A). Both parties are risk-neutral and have the same discount rate $r > 0$. Time is continuous and indexed by $t \in [0, \infty)$. The persistent type θ is drawn and learned by the agent at the initial date $t = 0$. The principal does not know θ. It is common knowledge that θ is a random draw from the uniform distribution over $\Theta = \left[\underline{\theta}, \overline{\theta}\right]$, where $0 \leq \underline{\theta} < \bar{\theta}$. Without loss of generality, we normalize $\bar{\theta} = 1$. In Section 6.2, we generalize our analysis to non-uniform distributions.

We focus on the case of a call option. We will refer to it as the option to invest, but it can

[4] Ottaviani and Sorensen (2006a,b) study a single-period reputational cheap talk setting, where the expert is concerned about appearing well-informed. Boot, Milbourn, and Thakor (2005) compare delegation and centralization when the agent's reputational concerns can distort her recommendations on whether to accept the project.

[5] Ely (2015) analyzes a setting with stochastically changing private information, where the informed party can commit to an information policy that shapes the beliefs of the uninformed party.

[6] Grenadier and Malenko (2011), Morellec and Schuerhoff (2011), Bustamante (2012), Grenadier, Malenko, and Strebulaev (2014).

capture any perpetual American call option, such as the option to do an IPO or to launch a new product. We also extend the analysis to a put option (e.g., if the decision is about shutting down a plant) and show that the main results continue to hold (see Section C of the Online Appendix).

The exercise at time t generates the payoff to the principal of $\theta X(t) - I$, where $I > 0$ is the exercise price (the investment cost), and $X(t)$ follows geometric Brownian motion with drift α and volatility σ:[7]

$$dX(t) = \alpha X(t)\,dt + \sigma X(t)\,dW(t),$$

where $\sigma > 0$, $r > \alpha$, and $dW(t)$ is the increment of a standard Wiener process.[8] We assume that the starting point $X(0)$ is low enough.[9] Process $X(t)$, $t \geq 0$ is observable by both the principal and the agent. As an example, consider an oil-producing firm that owns an oil well and needs to choose the optimal time to begin drilling. The publicly observable oil price process is represented by $X(t)$. The top management of the firm has authority over the decision to drill. The regional manager has private information about how much oil the well contains (θ), which stems from her local knowledge and prior experience with neighboring wells.

While the agent knows θ, he is biased. Specifically, upon exercise, the agent receives the payoff of $\theta X(t) - I + b$, where $b \neq 0$ is his commonly known bias. Positive bias $b > 0$ means that the agent is biased towards early exercise: his personal exercise price $(I - b)$ is lower than the principal's (I), so his most preferred timing of exercise is earlier than the principal's for any θ. Similarly, negative bias $b < 0$ means that the agent favors late exercise. These preferences can be viewed as reduced-form implications of an existing revenue-sharing agreement.[10] An alternative way to model the conflict of interest is to assume that $b = 0$ but the players discount the future using different discount rates. An early exercise bias corresponds to the agent being more impatient than the principal, $r_A > r_P$, and vice versa. We have analyzed the setting with different discount rates and shown that the results are identical to those in the bias setting (see Section C of the

[7] To illustrate the intuition behind our results, we also analyze a very simple example without any stochastic structure in Section A of the Appendix. This example is similar to the quadratic-uniform setting in Crawford and Sobel (1982).

[8] Our results also hold if $\sigma = 0$ and $\alpha > 0$, i.e., when the state increases deterministically with time. If $\sigma > 0$, the sign of the drift is not important for the qualitative results.

[9] Specifically, $X(0) < \min(X_P^*(1), X_A^*(1))$, where $X_P^*(\theta)$ and $X_A^*(\theta)$ are, respectively, the optimal exercise thresholds of the principal and the agent defined below. This assumption guarantees that there is disagreement between the two parties over the timing of exercise and that immediate exercise does not happen.

[10] For example, suppose that the principal supplies financial capital $\hat{I}$, the agent supplies human capital ("effort") valued at $\hat{e}$, and the principal and the agent hold fractions α_P and α_A of equity of the realized value from the project. Then, at exercise, the principal's (agent's) expected payoff is $\alpha_P \theta X(t) - \hat{I}$ $(\alpha_A \theta X(t) - \hat{e})$. This is analogous to the specification in the model with $I = \frac{\hat{I}}{\alpha_P}$ and $b = \frac{\hat{I}}{\alpha_P} - \frac{\hat{e}}{\alpha_A}$.

Online Appendix).

Following most of the literature on delegation, we do not allow the principal to make contingent transfers to the agent. In practice, decision-making inside firms mostly occurs via the allocation of control rights and informal communication, and hence it is important to study such settings. A plausible rationale for this is that the allocation of control rights is a simple solution to the problem of complexity of contracts with contingent transfers. Indeed, agents in organizations usually make many decisions, and writing complex contracts that specify transfers for all decisions and all possible outcomes of each decision is prohibitively costly. Furthermore, in some organizational settings, such as in government, transfers are explicitly ruled out by law.[11]

2.1 Optimal exercise policy for the principal and agent

Before presenting the main analysis, we consider two simple settings: one in which the principal knows θ and the other in which the agent has formal authority to exercise the option.

Optimal exercise policy for the principal. Suppose that the principal knows θ, so communication with the agent is irrelevant. In the Online Appendix, we show that following the standard arguments (e.g., Dixit and Pindyck, 1994), the optimal strategy for type θ is to exercise the option when $X(t)$ reaches threshold $X_P^*(\theta)$, where

$$X_P^*(\theta) = \frac{\beta}{\beta-1}\frac{I}{\theta} \tag{1}$$

and $\beta > 1$ is the positive root of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0$. The value of the option to the principal if the current value of $X(t)$ is X satisfies

$$V_P^*(X,\theta) = \begin{cases} \left(\frac{X}{X_P^*(\theta)}\right)^\beta \left(\theta X_P^*(\theta) - I\right), & \text{if } X \leq X_P^*(\theta) \\ \theta X - I, & \text{if } X > X_P^*(\theta). \end{cases} \tag{2}$$

Optimal exercise policy for the agent. Suppose that the agent has formal authority over when to exercise the option. If $b < I$, then substituting $I - b$ for I in (1), the agent's optimal

[11] In Section C of the Online Appendix, we allow the principal to write simple compensation contracts, such as offering the agent a payment upon exercise (for the late exercise bias case) or a flow of payments until exercise (for the early exercise case). We show that the optimal compensation scheme of this type never eliminates the conflict between the agent and the principal, and hence the setting and implications of our paper are robust to allowing for simple compensation contracts.

exercise strategy is to exercise the option at the first moment when $X(t)$ exceeds the threshold

$$X_A^*(\theta) = \frac{\beta}{\beta - 1}\frac{I - b}{\theta}. \tag{3}$$

If $b \geq I$, the optimal exercise strategy for the agent is to exercise the option immediately.

3 Optimal mechanism with commitment

To compare centralized decision-making with delegation, it will be useful to solve for the optimal decision-making rule if the principal has commitment power. In this section, we characterize the optimal mechanism in the class of threshold-exercise policies. A policy is called threshold-exercise if for every type $\theta \in \Theta$, there exists a threshold $\hat{X}(\theta)$, such that the option is exercised when $X(t)$ reaches $\hat{X}(\theta)$ for the first time.[12]

By the revelation principle, we can restrict attention to direct revelation mechanisms, i.e., those in which the message space is $\Theta = [\underline{\theta}, 1]$ and that provide the agent with incentives to report his type θ truthfully. Hence, we consider mechanisms of the form $\{\hat{X}(\theta) \geq X(0), \theta \in \Theta\}$: if the agent reports θ, the principal exercises when $X(t)$ first passes threshold $\hat{X}(\theta)$. Let $\hat{U}_A(\hat{X}, \theta)$ and $\hat{U}_P(\hat{X}, \theta)$ denote the time-zero expected payoffs of the agent and the principal, respectively, when type is θ and the exercise occurs at threshold $\hat{X}$. The optimal mechanism maximizes the principal's expected payoff subject to the agent's IC constraint:

$$\max_{\{\hat{X}(\theta), \theta \in \Theta\}} \int_{\underline{\theta}}^{1} \hat{U}_P(\hat{X}(\theta), \theta) \frac{1}{1 - \underline{\theta}} d\theta \tag{4}$$

$$s.t.\ \hat{U}_A(\hat{X}(\theta), \theta) \geq \hat{U}_A(\hat{X}(\hat{\theta}), \theta)\ \forall \theta, \hat{\theta} \in \Theta. \tag{5}$$

The next result characterizes the optimal threshold-exercise decision-making rule.

Lemma 1. *The optimal incentive-compatible threshold schedule $\hat{X}(\theta)$, $\theta \in \Theta$, is given by:*

- *If $b \in \left(-\infty, -\frac{1-\underline{\theta}}{1+\underline{\theta}} I\right] \cup \left[\frac{1-\underline{\theta}}{1+\underline{\theta}} I, \infty\right)$, then $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$ for any $\theta \in \Theta$.*

[12] We restrict attention to mechanisms with threshold exercise because the goal of this analysis is to provide a benchmark to compare delegation and centralized decision-making, and both delegation and centralization feature threshold exercise. The solution for the optimal mechanism in a more general class of mechanisms, in particular, those that allow for randomization, is beyond the scope of the paper.

- *If* $b \in \left(-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0\right]$, *then* $\hat{X}(\theta) = \begin{cases} \frac{\beta}{\beta-1}\frac{I+b}{\underline{\theta}}, & \text{if } \theta < \frac{I-b}{I+b}\underline{\theta}; \\ \frac{\beta}{\beta-1}\frac{I-b}{\theta}, & \text{if } \theta \geq \frac{I-b}{I+b}\underline{\theta}. \end{cases}$

- *If* $b \in \left[0, \frac{1-\underline{\theta}}{1+\underline{\theta}}I\right)$, *then* $\hat{X}(\theta) = \begin{cases} \frac{\beta}{\beta-1}\frac{I-b}{\theta}, & \text{if } \theta < \frac{I-b}{I+b}; \\ \frac{\beta}{\beta-1}(I+b), & \text{if } \theta \geq \frac{I-b}{I+b}. \end{cases}$

The lemma shows that the optimal threshold-exercise mechanism is interval delegation: the principal lets the agent choose any exercise threshold within a certain interval. The optimal decision rule features perfect separation of types up to a cutoff and pooling beyond the cutoff. The reasoning behind this result is similar to the reasoning of why the optimal decision rule is interval delegation in many static problems (Melumad and Shibano, 1991; Alonso and Matouschek, 2008; Amador and Bagwell, 2013). Intuitively, because the agent does not receive additional private information over time and the optimal stopping rule can be summarized by a threshold, the optimal dynamic contract is similar to the optimal contract in a static game with equivalent payoff functions.

Having derived the optimal full-commitment decision rule, we next analyze under what circumstances the principal can implement this optimal decision rule without full commitment power.

4 Centralized decision-making with communication

In this section, we consider the case of no commitment power, where the principal can only engage in cheap talk communication with the agent while retaining formal authority over the decision. This problem is the option exercise analogue of Crawford and Sobel's (1982) static cheap talk model.

4.1 Timing and equilibrium notion

The timing is as follows. At each time t, knowing the type $\theta \in \Theta$ and the history of the game $\mathcal{H}_t$, the agent decides on a message $m(t) \in M$ to send to the principal, where M is a set of messages. At each t, the principal decides whether to exercise the option or not, given $\mathcal{H}_t$ and the current message $m(t)$. That is, the agent's and the principal's strategies are, respectively, $m_t : \Theta \times \mathcal{H}_t \to M$ and $e_t : \mathcal{H}_t \times M \to \{0, 1\}$, where $e_t = 1$ stands for "exercise" and $e_t = 0$ stands for "wait." Let $\tau(e) \equiv \inf\{t : e_t = 1\}$ denote the stopping time implied by strategy e of the principal. Finally, let $\mu(\theta|\mathcal{H}_t)$ and $\mu(\theta|\mathcal{H}_t, m(t))$ denote the updated probability that the principal assigns to the type of the agent being θ given the history $\mathcal{H}_t$ before and after getting message $m(t)$, respectively.

We focus on equilibria in pure strategies. The equilibrium concept is *Perfect Bayesian Equilibrium in Markov strategies (PBEM)*, which requires that the agent's and the principal's strategies are sequentially optimal, beliefs are updated according to Bayes' rule whenever possible, and the equilibrium strategies are Markov. In particular, the Markov property requires that the players' strategies are only functions of the payoff-relevant information at any time t, i.e., the type of the agent, the current value of the state process $X(t)$, and the principal's beliefs about the agent's type. The formal definition of the PBEM is presented in Section A of the Online Appendix.

Bayes' rule does not apply to messages that are not sent by any type in equilibrium. To restrict beliefs following such off-equilibrium messages, we make the following assumption.

Assumption 1. *If at any t, the principal's belief $\mu(\cdot|\mathcal{H}_t)$ and the observed message $m(t)$ are such that no type that could exist (according to the belief $\mu(\cdot|\mathcal{H}_t)$) could send $m(t)$, then the belief is unchanged.*

This assumption is related to a frequently imposed restriction in models with two types that if, at any point, the posterior assigns probability one to a given type, then this belief persists no matter what happens (e.g., Rubinstein, 1985; Halac, 2012). Because our model features a continuum of types, an action that no one was supposed to take may occur off equilibrium even if the belief is not degenerate. As a consequence, we impose a stronger restriction.

Let stopping time $\tau^*(\theta)$ denote the equilibrium exercise time if the type is θ. In almost all standard option exercise models, the optimal exercise strategy for a perpetual American call option is a threshold: it is optimal to exercise at the first instant the state process $X(t)$ exceeds some critical level. It is thus natural to look for equilibria that exhibit a similar property, formally defined as:

Definition 1. *An equilibrium is a **threshold-exercise** PBEM if for all $\theta \in \Theta$, $\tau^*(\theta) = \inf\left\{t \geq 0 | X(t) \geq \bar{X}(\theta)\right\}$ for some $\bar{X}(\theta)$ (possibly infinite).*

For any threshold-exercise equilibrium, we denote the set of equilibrium exercise thresholds by

$\mathcal{X} \equiv \{X : \exists \theta \in \Theta \text{ such that } \bar{X}(\theta) = X\}$. In the Online Appendix, we show that any threshold-exercise equilibrium has the following two properties.

First, the option is exercised weakly later if the agent has less favorable information: $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ whenever $\theta_2 \geq \theta_1$. Intuitively, because talk is "cheap," the agent of type θ_1 can adopt the message strategy of type $\theta_2 > \theta_1$, and vice versa. Thus, when choosing between communication strategies that induce exercise at thresholds $\bar{X}(\theta_1)$ and $\bar{X}(\theta_2)$, type θ_1 must prefer the former, and type θ_2 must prefer the latter. This is simultaneously possible only if $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$.

Second, it is without loss of generality to reduce the message space to binary messages. Intuitively, at each time the principal faces a binary decision: to exercise or to wait. Because the agent's information is important only for the timing of the exercise, one can achieve the same efficiency by choosing the timing of communicating a binary message as through the richness of the message space. Therefore, message spaces that are richer than binary cannot improve efficiency. Specifically, we show that for any threshold-exercise equilibrium, there exists an equilibrium with a binary message space $M = \{0, 1\}$ that implements the same exercise times and hence features the same payoffs of both players and takes the following simple form. At any time t, the agent can send one of two messages, 1 ("exercise") or 0 ("wait"). The agent recommends exercise if and only if $X(t) \geq \bar{X}(\theta)$. The principal also plays a threshold strategy: If she believes that $\theta \in [\check{\theta}_t, \hat{\theta}_t]$, she exercises the option if and only if $X(t) \geq \check{X}(\check{\theta}_t, \hat{\theta}_t)$. As a consequence of the agent's strategy, there is a set $\mathcal{T}$ of "informative" times, when the agent's message has information content, i.e., it affects the belief of the principal and, in turn, her exercise decision. These are instances when $X(t)$ first passes a new threshold from the set $\mathcal{X}$. At all other times, the agent's message has no information content. In equilibrium, type θ recommends exercise at the first time when $X(t)$ passes $\bar{X}(\theta)$ for the first time, and the principal responds by exercising immediately. In what follows, we focus on threshold-exercise PBEM of this form and refer to them as simply "equilibria."

4.2 When does centralization implement the optimal mechanism?

We now examine under what conditions the optimal full-commitment mechanism can be implemented with no commitment power of the principal. In other words, when does the communication game described above have an equilibrium that features the exercise policy from Lemma 1? We show that the answer crucially depends on whether the agent is biased towards late or early exercise. Specifically:

Proposition 1.

1. *If $b < 0$, there always exists an equilibrium of the communication game that implements the optimal mechanism from Lemma 1. This equilibrium is as follows:*

   If $b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the equilibrium is babbling and the principal exercises at the uninformed threshold $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. If $b \in (-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0]$, there exists a cutoff X^*, potentially infinite, such that the principal's strategy is: (1) to wait if the agent sends message $m = 0$ and to exercise at the first time t at which the agent sends message $m = 1$, provided that $X(t) \in [X_A^*(1), X^*]$ and $X(t) = \max_{s \leq t} X(s)$; (2) to exercise at the first time t at which $X(t) \geq X^*$, regardless of the agent's message. The agent of type θ sends $m = 1$ at the first moment when $X(t)$ crosses the minimum between his most-preferred threshold $X_A^*(\theta)$ and X^*, and sends message $m = 0$ before that. Threshold X^* is given by

$$X^* = \begin{cases} \frac{\beta}{\beta-1}\frac{I+b}{\underline{\theta}} = X_A^*(\frac{I-b}{I+b}\underline{\theta}) & \text{if } \underline{\theta} > 0, \\ \infty & \text{if } \underline{\theta} = 0, \end{cases}$$

   where $\hat{\theta}^* \equiv \frac{I-b}{I+b}\underline{\theta} < 1$.

2. *If $b \in (0, \frac{1-\underline{\theta}}{1+\underline{\theta}}I)$, there is no equilibrium that implements the optimal mechanism with commitment. If $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the babbling equilibrium where the principal exercises at the uninformed threshold $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$ implements the optimal mechanism.*

First, consider the case of an agent biased towards late exercise. Similarly to the optimal mechanism in Lemma 1, the equilibrium in Proposition 1 features full separation of types above $\hat{\theta}^*$ (with exercise at the agent's preferred threshold) and pooling of types below $\hat{\theta}^*$. The intuition behind this equilibrium is as follows. When the agent with a late exercise bias recommends exercise at his preferred threshold, the principal learns that it is too late to do so and is tempted to go back in time and exercise the option in the past. This, however, is not feasible, and hence the principal finds it optimal to follow the agent's recommendation. Knowing that, the agent communicates honestly, but communication occurs with delay. At any time before receiving the recommendation to exercise, the principal faces the following trade-off. On the one hand, she can wait and see what the agent will recommend in the future. This leads to more informative exercise because the agent communicates his information, but has a drawback in that exercise will be delayed. On the other hand, the principal can disregard the agent's future recommendations and exercise immediately. This results in less informative exercise, but not in excessive delay. Thus, the trade-off is between the value of information and the cost of delay. When the agent's bias is very large, $b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the cost of delay is too high and induces the principal to exercise at her uninformed threshold without waiting for the agent's recommendation. When the agent's bias is moderate, $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the cost of delay is not too high and some communication occurs. As time goes by and the agent continues recommending against exercise, the principal learns that θ is not too high (below some cutoff $\hat{\theta}_t$ at time t), and the interval $[\underline{\theta}, \hat{\theta}_t]$, which captures the principal's posterior belief, shrinks over time. For any $\underline{\theta} > 0$, the shrinkage of this interval implies that the remaining value of the agent's information declines over time. Once the interval shrinks to $[\underline{\theta}, \hat{\theta}^*]$, which happens at threshold X^*, the remaining value of the agent's information becomes sufficiently small, so the principal finds it optimal to exercise immediately. The comparative statics of the cutoff type $\hat{\theta}^*$ are intuitive: As b decreases, i.e., the conflict of interest gets bigger, $\hat{\theta}^*$ increases and X^* decreases, implying that the principal waits less for the agent's recommendation. The red line in Figure 1 illustrates the exercise threshold in this equilibrium for parameters $\underline{\theta} = 0.15$, $r = 0.15$,[13] $\alpha = 0.05$, $\sigma = 0.2$, $I = 1$, and $b = -0.25$.

In contrast, if the agent is biased towards early exercise, the optimal commitment mechanism generally cannot be implemented through centralized decision-making. This asymmetry occurs because of the asymmetric nature of time: the set of choices that the principal has (when to exercise) shrinks over time. When the agent is biased towards late exercise, then even without formal commitment power, as time passes, the principal effectively commits not to exercise earlier because she cannot go back in time. In contrast, no such commitment power exists in the case of an early exercise bias: If the agent follows the strategy of recommending exercise at his preferred threshold $X_A^*(\theta)$, the principal infers the agent's type perfectly and prefers to delay exercise upon getting the recommendation to exercise. Knowing this, the agent is tempted to change his recommendation strategy, mimicking a higher type. Thus, no equilibrium that features separation of types exists in this case. Because the optimal mechanism for any $b < \frac{1-\underline{\theta}}{1+\underline{\theta}}I$ features separation of types over some interval, it cannot be implemented.

Note also that a special case of the equilibrium in Proposition 1 is $\underline{\theta} = 0$. As long as the agent's bias is not very high, $b \geq -I$, there is full information revelation, but communication and exercise are inefficiently (from the principal's perspective) delayed. Using the terminology of Aghion and

[13] The discount rate 0.15 can be interpreted as the sum of the risk-free interest rate 0.05 and the intensity 0.1 with which the investment opportunity disappears.

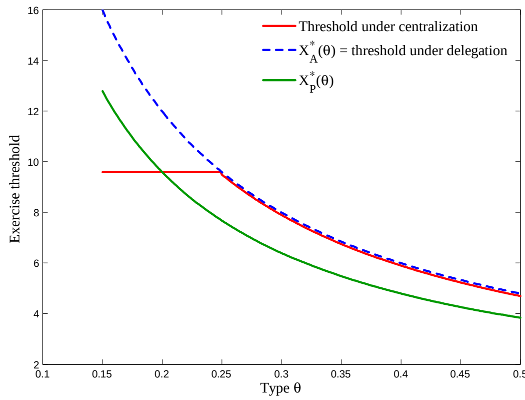

Figure 1. Equilibrium for the case $\underline{\theta} > 0$, $b < 0$. The figure presents the equilibrium with continuous exercise up to a cutoff for parameters $\underline{\theta} = 0.15$, $r = 0.15$, $\alpha = 0.05$, $\sigma = 0.2$, $I = 1$, and $b = -0.25$.

Tirole (1997), the equilibrium features unlimited real authority of the agent, even though the principal has unlimited formal authority. The reason why this equilibrium is equivalent to full delegation, rather than delegation up to a finite cutoff, is that when $\underline{\theta} = 0$, the problem exhibits stationarity in the following sense. Because the prior distribution of types is uniform over $[0, 1]$ and the payoff structure is multiplicative, a time-t sub-game in which the principal's posterior belief is uniform over $[0, \hat{\theta}]$ is equivalent to the game where the belief is that θ is uniform over $[0, 1]$, the true type is $\frac{\theta}{\hat{\theta}}$, and the modified state process is $\hat{\theta} X(t)$. Because of stationarity, the trade-off between the value of information and the cost of delay persists over time even though the principal updates her belief: as long as the agent's bias is not too high ($b > -I$), the principal finds it optimal to wait for the agent's recommendation for any current belief.

4.3 Equilibria in the stationary case

The stationary nature of the game for $\underline{\theta} = 0$ makes the model very tractable and allows us to fully characterize equilibria of the dynamic communication game. It is natural to restrict attention to *stationary* equilibria, which are defined as follows.

Definition 2. *Suppose $\underline{\theta} = 0$. An equilibrium (m^*, e^*, μ^*, M) is **stationary** if whenever posterior belief $\mu^*(\cdot|\mathcal{H}_t)$ is uniform over $[0, \hat{\theta}]$ for some $\hat{\theta} \in (0, 1)$, then for all $\theta \in [0, \hat{\theta}]$:*

$$m^*(\theta, X(t), \mu^*(\cdot|\mathcal{H}_t)) = m^*\left(\frac{\theta}{\hat{\theta}}, \hat{\theta} X(t), \mu^*(\cdot|\mathcal{H}_0)\right), \quad (6)$$

$$e^*(X(t), \mu^*(\cdot|\mathcal{H}_t, m(t))) = e^*\left(\hat{\theta} X(t), \mu^*(\cdot|\mathcal{H}_0, m(t))\right). \quad (7)$$

Condition (6) means that the message of type $\theta \in [0, \hat{\theta}]$ when the public state is $X(t)$ and the posterior is uniform over $[0, \hat{\theta}]$ is the same as the message of type $\frac{\theta}{\hat{\theta}}$ when the public state is $\hat{\theta}X(t)$ and the posterior is uniform over $[0, 1]$. Condition (7) means that the exercise strategy of the principal is the same when the public state is $X(t)$ and her belief is that θ is uniform over $[0, \hat{\theta}]$ as when the public state is $\hat{\theta}X(t)$ and her belief is that θ is uniform over $[0, 1]$.

Stationarity and the property $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ for $\theta_2 \geq \theta_1$ imply that any stationary equilibrium must take one of two forms.[14] The first is an equilibrium that features continuous exercise at the agent's optimal threshold $X_A^*(\theta)$, i.e., the equilibrium characterized in the first part of Proposition 1. The second are equilibria that have a partition structure, with the set of types partitioned into intervals and each interval inducing exercise at a given threshold. Moreover, stationarity implies that the set of partitions must be infinite and take the form $[\omega, 1]$, $[\omega^2, \omega]$, ..., $[\omega^n, \omega^{n-1}]$, ..., $n \in N$, for some $\omega \in [0, 1)$, where $\mathbb{N}$ is the set of natural numbers. This implies that the set of exercise thresholds $\mathcal{X}$ is given by $\left\{\bar{X}, \frac{\bar{X}}{\omega}, \frac{\bar{X}}{\omega^2}, ..., \frac{\bar{X}}{\omega^n}, ...\right\}$, $n \in \mathbb{N}$, for some $\bar{X} > 0$, such that if $\theta \in (\omega^n, \omega^{n-1})$, the option is exercised at threshold $\frac{\bar{X}}{\omega^{n-1}}$. We refer to an equilibrium of this form as a ω-*equilibrium.*

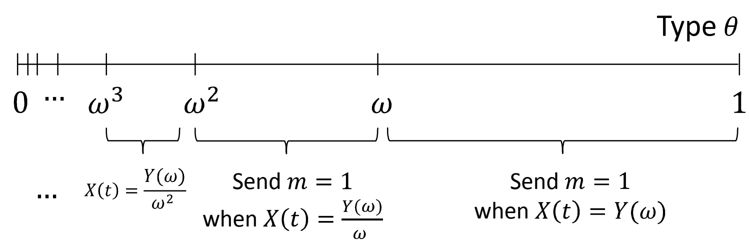

Figure 2. Partitions in a ω-equilibrium.

For ω and $Y(\omega)$ to constitute an equilibrium, the incentive compatibility (IC) conditions for the principal and the agent must hold. Pair $(\omega, \bar{X})$ satisfies the agent's IC condition only if types above ω have incentives to recommend exercise ($m = 1$) at threshold $\bar{X}$ rather than to wait, whereas types below ω have incentives to recommend delay ($m = 0$). The proof of Proposition 2 shows that the agent's IC condition holds if and only if type ω is exactly indifferent between exercising the option at threshold $\bar{X}$ and at threshold $\frac{\bar{X}}{\omega}$, and this indifference condition reduces

[14]The argument is as follows. If there is separation (pooling) of types between some cutoff $\hat{\theta}$ and 1, there must also be separation (pooling) of types between $\hat{\theta}^2$ and $\hat{\theta}$. Iterating this argument implies that either all types separate or there is a sequence of partitions, each being a multiple of the previous one.

to the constraint

$$\bar{X} = Y(\omega) \equiv \frac{\left(1-\omega^{\beta}\right)(I-b)}{\omega\left(1-\omega^{\beta-1}\right)}. \tag{8}$$

The partitions in a ω-equilibrium are illustrated in Figure 2.

Next, consider the principal's problem. For ω and $\bar{X}$ to constitute an equilibrium, the principal must have incentives: (1) to exercise the option immediately when the agent sends message $m = 1$ at a threshold in $\mathcal{X}$; and (2) not to exercise the option before getting message $m = 1$. We refer to the former (latter) IC condition as the *ex-post (ex-ante) IC constraint.* The proof of Proposition 2 shows that the ex-post IC condition for the principal is equivalent to

$$Y(\omega) \geq \frac{\beta}{\beta-1}\frac{2I}{\omega+1}. \tag{9}$$

This condition has a clear intuition. Suppose that $X(t)$ reaches threshold $\bar{X} = Y(\omega)$ for the first time, and the principal receives recommendation $m = 1$ at that instant. By Bayes' rule, the principal updates her beliefs to θ being uniform on $[\omega, 1]$. Condition (9) ensures that the current value of the state process, $Y(\omega)$, exceeds the optimal exercise threshold of the principal given these updated beliefs, $\frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, and hence the principal finds it optimal to exercise immediately.

We show that when $b < 0$, the principal's ex-post IC condition (9) is satisfied for any $\omega \in (0, 1)$. Intuitively, this is because the agent is biased towards late exercise, and hence the principal does not benefit from further delay. In contrast, when the agent favors early exercise, (9) is satisfied if and only if ω is low enough. The intuition why the ex-post IC condition is violated if ω is large is similar to the standard intuition of why sufficiently efficient information revelation is impossible in cheap talk games: Because the agent has an early exercise bias and the principal can wait and exercise later after getting the agent's message to exercise, the agent's message cannot be too informative about his type. Formally, we show that for any $b \in (0, I)$, there exists a unique $\omega^* \in (0, 1)$ such that (9) is satisfied as an equality and that the principal's ex-post IC condition is satisfied if and only if $\omega \leq \omega^*$.

Finally, the principal's ex-ante IC condition is satisfied if and only if communication is informative enough, which puts a lower bound on ω, denoted $\underline{\omega} > 0$. The set of equilibria with partitioned exercise is illustrated in Figure 3.

The following proposition summarizes the set of all stationary equilibria.

Proposition 2. *If $b \in [-I, I)$, the set of non-babbling stationary equilibria is given by:*

- *Equilibria with partitioned exercise (ω-equilibria) exist if and only if $b \in (-I, I)$. If $b \in (-I, 0)$, there exists a unique ω-equilibrium for each $\omega \in [\underline{\omega}, 1)$, and if $b \in (0, I)$, there exists a unique ω-equilibrium for each $\omega \in [\underline{\omega}, \omega^*]$, where $0 < \underline{\omega} < \omega^* < 1$, ω^* is the unique solution to $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, and $\underline{\omega}$ is uniquely defined by the condition that the principal's ex-ante IC constraint is binding. In the ω-equilibrium, the principal exercises at time t at which $X(t)$ crosses threshold $Y(\omega)$, $\frac{1}{\omega}Y(\omega)$, ... for the first time, provided that the agent sends message $m = 1$ at that point, where $Y(\omega)$ is given by (8). The principal does not exercise the option at any other time. The agent of type θ sends $m = 1$ at the first moment when $X(t)$ crosses threshold $\frac{1}{\omega^n}Y(\omega)$, where $n \geq 0$ is such that $\theta \in (\omega^{n+1}, \omega^n)$.*

- *Equilibrium with continuous exercise exists if and only if $b \in [-I, 0)$. The principal exercises at the first time t at which the agent sends $m = 1$, provided that $X(t) \geq X_A^*(1)$ and $X(t) = \max_{s \leq t} X(s)$. The agent of type θ sends $m = 1$ at the first moment when $X(t)$ crosses his most-preferred threshold $X_A^*(\theta)$.*

If $b < -I$ or $b \geq I$, the only stationary equilibrium is babbling, where the principal exercises the option at her optimal uninformed threshold $\bar{X}_u = \frac{\beta}{\beta-1}2I$.

Clearly, not all of these equilibria are equally reasonable. It is common in cheap talk games to focus on the equilibrium with the most information revelation, which corresponds to the equilibrium with continuous exercise for $b < 0$ and the ω^*-equilibrium for $b > 0$.[15] It turns out that these equilibria dominate other equilibria in the following sense.

Proposition 3. *If $b < 0$, the equilibrium with continuous exercise from Proposition 2 dominates all other possible equilibria in the Pareto sense: both the agent's expected payoff for each realization of θ and the principal's expected payoff are higher in this equilibrium than in any other equilibrium. If $b > 0$, the ω^*-equilibrium dominates other stationary equilibria with partitioned exercise in the following sense: both the principal's expected payoff and the ex-ante expected payoff of the agent before θ is realized are higher in the ω^*-equilibrium than in the ω-equilibrium for any $\omega < \omega^*$.*

[15] In general, equilibrium selection in cheap talk games is a delicate issue. Unfortunately, most equilibrium refinements that reduce the set of equilibria in costly signaling games do not work well in games of costless signaling (i.e., cheap talk). Some formal approaches to equilibrium selection in cheap talk games are provided by Farrell (1993) and Chen, Kartik, and Sobel (2008).

Intuitively, when $b < 0$, the equilibrium with continuous exercise both implements the optimal mechanism for the principal and ensures that exercise occurs at the unconstrained optimal time of any type θ of the agent. When $b > 0$, the ω^*-equilibrium is the only equilibrium in which exercise is unbiased: since the principal's ex-post IC condition holds as an equality, the exercise rule maximizes the principal's payoff given that the agent's type lies in a given partition. In all other equilibria, there is both loss of information and delay in option exercise, which is detrimental for both the principal and the agent with a bias towards early exercise. Interestingly, delay in exercise in these equilibria occurs despite the fact that the agent is biased towards early exercise.

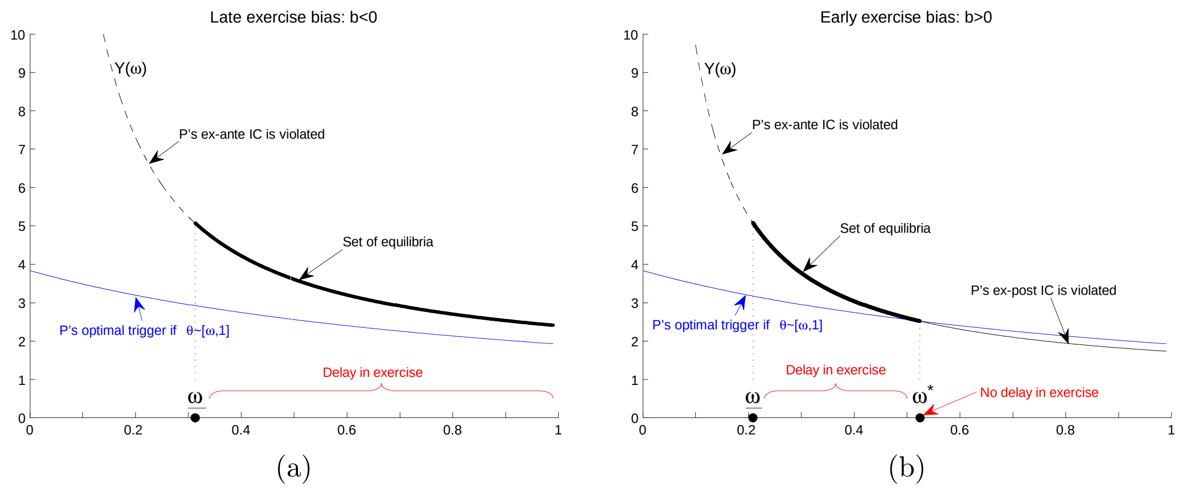

Figure 3. Equilibria with partitioned exercise. The figures present the partition equilibria for $\underline{\theta} = 0$, $r = 0.15$, $\alpha = 0.05$, $\sigma = 0.2$, and $I = 1$. The agent's bias is $b = -0.25$ in figure (a) and $b = 0.1$ in figure (b). In both figures, the black line represents the agent's IC condition, i.e., the function $Y(\omega)$, and the blue line represents the function $\frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, i.e., the principal's optimal exercise trigger if θ is uniform on $[\omega, 1]$.

Focusing on the most informative (ω^*) equilibrium in the early exercise bias case, the informativeness of communication exhibits interesting comparative statics.

Proposition 4. *Consider the case of an agent biased towards early exercise, $b > 0$. Then, ω^* decreases in b and increases in β, and hence decreases in σ and α, and increases in r.*

The result that ω^* decreases in the agent's bias is in line with the result of Crawford and Sobel (1982) that less information is revealed if the misalignment of preferences is bigger. More interesting are the comparative statics results in σ, α, and r. The proposition shows that communication is less efficient when the option to wait is more valuable. For example, there is less information revelation (ω^* is lower) if the environment is more uncertain (σ is higher). Intuitively, higher uncertainty increases the value of the option to delay exercise and thus effectively increases the conflict of interest between the principal and the agent biased towards early exercise. Similarly, communication is less efficient in lower interest rate and higher growth rate environments.

4.3.1 Comparison with static communication

It is instructive to highlight the role of dynamic communication by comparing the above equilibria to those in the benchmark model, where communication is static and restricted to a one-shot interaction at the beginning of the game. Specifically, consider a restricted version of the model, where instead of communicating with the principal continuously, the agent sends a single message at time $t = 0$ and there is no subsequent communication. After receiving the message, the principal updates her beliefs about θ and exercises the option at the optimal threshold given these beliefs. This problem is closest to the cheap talk models with a multiplicative bias, studied by Melumad and Shibano (1991) and Alonso (2009).[16] In these models, when the agent wants a higher loading of the action on θ (the early exercise bias case in our paper), there is an equilibrium with an infinite number of partitions.[17] Intuitively, because the agent has a bias for earlier exercise, communication is more credible when the agent sends messages recommending later exercise, i.e., when type θ is lower. In the limit, as $\theta \to 0$, communication becomes fully informative because the interests of the agent and the principal coincide when $\theta = 0$: both prefer to never exercise the option. In contrast, when the agent wants a smaller loading of the action on θ (the late exercise bias case in our paper), equilibria have a finite number of partitions (Alonso, 2009). Intuitively, communication is less credible when the agent recommends later exercise, which puts a limit on the number of partitions.

The similarity between the static communication and the dynamic communication problems is that both lead to implications that are asymmetric in the sign of the bias. The difference, however, is that the implications are the opposite. To the extent that the number of partitions measures the effectiveness of communication, communication in the static problem is more efficient if the agent

[16] To see the mapping more clearly, let "action" d be defined as the inverse of the exercise threshold. Then, the agent's most preferred action is $d = \frac{\beta-1}{\beta(I-b)}\theta$, and the principal's optimal action given information set $\mathcal{I}$ is $d = \frac{\beta-1}{\beta I}\mathbb{E}[\theta|\mathcal{I}]$. Hence, the early exercise bias case, $b > 0$, corresponds to the agent who wants a higher loading of the action on θ. While similar, the static analogue of our model is not the same as in Melumad and Shibano (1991) and Alonso (2009), since the value function in our model is not quadratic. We thank Ricardo Alonso for pointing out this mapping.

[17] See also Gordon (2010) for more general conditions for existence of equilibria with an infinite number of partitions.

is biased towards earlier exercise than towards later exercise. In contrast, when communication occurs over time, communication is extremely efficient if the agent is biased towards later exercise. Intuitively, because the principal cannot go back in time, the set of actions that the principal can take (when to exercise) decreases over time. When the set of actions shrinks in the direction of the agent's bias, which happens if the agent has a bias for later exercise, this gives the principal commitment power to follow the agent's recommendation, making communication efficient. In contrast, when the set of actions shrinks against the direction of the agent's bias, which happens if the agent has a bias for earlier exercise, this does not help, since the principal prefers to delay exercise if the agent recommends to exercise the option at his preferred time.

Another way to see how dynamic communication differs from static communication is to study whether the stationary equilibria characterized in Proposition 2 have equivalent equilibria in the static communication game. The answer is provided in the following proposition.

Proposition 5. *If $b < 0$, there is no non-babbling stationary equilibrium of the dynamic communication game that is also an equilibrium of the static communication game. If $b > 0$, the only non-babbling stationary equilibrium of the dynamic communication game that is also an equilibrium of the static communication game is the ω^*-equilibrium.*

The intuition is as follows. All non-babbling stationary equilibria of the dynamic communication game for $b < 0$ feature delay relative to what the principal's optimal timing of exercise would have been ex ante, given the information she learns in equilibrium. In a dynamic communication game, this delay is feasible because the principal learns information with delay, after her optimal (conditional on this information) exercise time has passed. However, in a static communication game, this delay cannot be sustained: since the principal learns all the information at time zero, her exercise decision is always optimal given the available information.[18] By the same argument, the only sustainable equilibrium of the dynamic communication game for $b > 0$ is the one that features no delay relative to the principal's optimal threshold, i.e., the ω^*-equilibrium.

Thus, even though the agent's information is persistent, the ability to communicate dynamically is crucial when the agent has a bias for later exercise. Dynamic communication helps both players because it ensures that the principal follows the agent's recommendation and thereby makes communication very efficient. In particular, dynamic communication allows the principal

[18] Similarly, in the non-stationary case, the equilibrium with continuous exercise up to a cutoff, described in Proposition 1, does not exist in the static communication game either.

to implement the optimal mechanism, while static communication does not.

5 Delegation

Section 4 shows that when the agent is biased towards early exercise, the principal cannot implement the full-commitment optimal mechanism if she has no commitment power. We therefore next ask whether the principal can implement the optimal mechanism if she has some, albeit less than full, commitment power. In particular, we study the case where the principal can commit to the allocation of the decision rights: She can either delegate formal authority to exercise the option to the agent or keep formal authority and play the communication game analyzed in the previous section. First, we consider the problem studied by Dessein (2002) for static decisions, but focus on stopping time decisions. Next, we consider the problem when the principal can postpone the delegation decision.

It follows from Proposition 1 that implications for delegation are different between the "late exercise bias" and the "early exercise bias" cases.

5.1 Delegation when the agent has a preference for late exercise

First, consider the case of the late exercise bias. Because centralized decision-making implements the optimal commitment mechanism when $b < 0$, the principal is always weaker better off retaining control and getting advice from the agent rather than delegating the exercise decision. Moreover, while delegation and communication are equivalent if $\underline{\theta} = 0$, delegation is strictly inferior to communication if $\underline{\theta} > 0$: Not delegating the decision and playing the communication game implements *constrained* delegation (delegation up to a cutoff), while delegation implements *unconstrained* delegation. This result is illustrated in Figure 1 and summarized in the following corollary.

Corollary to Proposition 1. *If $b < 0$, the principal always weakly prefers retaining control and getting advice from the agent to delegating the exercise decision. The preference is strict if $\underline{\theta} > 0$. If $\underline{\theta} = 0$, retaining control and delegation are equivalent.*

This result contrasts with the implications for static decisions, such as choosing the scale of the project. Dessein (2002) shows that in the leading quadratic-uniform setting of Crawford and Sobel (1982), regardless of the direction of the agent's bias, delegation always dominates communication as long as the agent's bias is not too high so that at least some informative communication is possible. For general payoff functions, Dessein (2002) shows that delegation is optimal if the agent's bias is sufficiently small. In contrast, we show that if the agent favors late exercise, then regardless of the magnitude of his bias, the principal never wants to delegate decision-making authority to him. Intuitively, the inability to go back in time allows the principal to commit to follow the recommendations of the agent. This commitment role of time ensures that communication is sufficiently effective so that delegation has no further benefit.[19]

5.2 Delegation when the agent has a preference for early exercise

Next, consider the case of the early exercise bias. Because the optimal commitment mechanism in Lemma 1 features constrained delegation, simple delegation does not implement the optimal mechanism. However, differently from the case of a late exercise bias, simple delegation can be preferred to centralization if the agent's bias is low enough. The following proposition summarizes our findings.

Proposition 6. *Suppose $b > 0$, $\underline{\theta} = 0$, and consider the most informative equilibrium of the communication game, ω^*. There exist $\underline{b}$ and $\bar{b}$, such that the principal's expected value in the ω^*-equilibrium is lower than her expected value under delegation if $b < \underline{b}$, and is higher than under delegation if $b > \bar{b}$.*

The result that delegation is beneficial when the agent's bias is small enough is similar to the result of Dessein (2002) for static decisions and shows that Dessein's argument extends to stopping time decisions when the agent favors early exercise. Intuitively, the principal faces a trade-off: delegation leads to early exercise due to the agent's bias but uses the agent's information more efficiently. When the agent's bias is small enough, the cost from early exercise is smaller than the cost due to the loss of the agent's information, and hence delegation dominates.

[19] Note also that in our context, exercise occurs with delay even under centralization. This is different from Bolton and Farrell (1990), where centralization helps avoid inefficient delay caused by coordination problems between competing firms.

5.3 Optimal timing of delegation

In a dynamic setting, the principal does not need to delegate authority to the agent from the start: she may retain authority for some time and delegate later. In this section, we study whether timing delegation strategically may help the principal. In particular, consider the following game: The principal and the agent play the communication game of Section 4, but at any time, the principal may delegate decision-making authority to the agent. After authority is granted, the agent retains it until the end of the game and thus is free to choose when to exercise the option.

According to Proposition 1, if the agent favors late exercise, the communication equilibrium implements the optimal mechanism with commitment. Hence, the principal cannot do better with delegation than with keeping authority forever and communicating with the agent. In contrast, when the agent favors early exercise, simply communicating with the agent, as well as delegating the decision to the agent at the initial point in time, brings a lower payoff than under the optimal mechanism. However, the next result shows that for any $\underline{\theta} \geq 0$, the principal can implement the optimal mechanism by delegating the decision at the right time.

Proposition 7. *If $b > 0$, there exists the following equilibrium. The principal delegates authority to the agent at the first moment when $X(t)$ reaches the threshold $X_d \equiv \min(\frac{\beta(I+b)}{\beta-1}, \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1})$ and does not exercise the option before that. For any θ, the agent sends message $m = 0$ at any point before he is given authority. If $\theta \geq \frac{I-b}{I+b}$, the agent exercises the option immediately after he is given authority, and if $\theta \leq \frac{I-b}{I+b}$, the agent exercises the option when $X(t)$ first reaches his preferred exercise threshold $X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$. The exercise threshold in this equilibrium coincides with the optimal exercise threshold under commitment.*

Intuitively, timing delegation strategically ensures that the information of low types ($\theta \leq \frac{I-b}{I+b}$) is used efficiently, and that all types above $\frac{I-b}{I+b}$ exercise immediately at the time of delegation, exactly as in the optimal contract. The higher is the agent's bias, the later will delegation occur.

Overall, the analysis in this section implies that the direction of the conflict of interest is the key driver of the allocation of authority for timing decisions. If the agent favors late exercise, the principal should always retain control and rely on communication with the agent. In contrast, if the agent favors early exercise, it is optimal to delegate the decision to the agent at some point in time.

6 Extensions

We develop two extensions of the basic model. First, we consider a setting in which at any point in time, the principal might learn the agent's information θ with some probability, even without any communication from the agent. We show that in the delay bias case ($b < 0$), centralized decision-making, where the principal keeps authority and communicates with the agent, strictly dominates both unconstrained and any constrained delegation. Second, we relax the assumption that the distribution of types is uniform and extend our main results to a large class of distributions.

6.1 Arrival of news about the project

Consider the model in the delay bias case with the following change. Suppose that there is a Poisson news arrival process that reveals type θ to the principal upon arrival. The arrival rate is $\lambda > 0$. In this setting, the equilibrium under centralized decision-making with communication takes the following form:

Proposition 8. *Consider centralized decision-making. If $b \in (-\frac{1-\theta}{1+\underline{\theta}}I, 0)$, there exists the following equilibrium. The principal's strategy after the arrival of the news is to exercise the option at the first time t at which $X(t) \geq X_P^*(\theta)$. The principal's strategy prior to the arrival of the news is: (1) to wait if the agent sends message $m = 0$ and to exercise at the first time t at which the agent sends message $m = 1$, provided that $X(t) \in [\tilde{X}_A(1), \check{X}]$ and $X(t) = \max_{s \leq t} X(s)$; (2) to exercise at the first time t at which $X(t) \geq \check{X}$, regardless of the agent's message, where threshold $\check{X}$ and function $\tilde{X}_A(\theta)$ are defined in the Appendix. The agent's strategy is to send $m = 1$ at the first moment when $X(t)$ crosses $\tilde{X}_A(\theta)$ and to send $m = 0$ before that. Furthermore, $X_P^*(\theta) < \tilde{X}_A(\theta) < X_A^*(\theta)$.*

Proposition 8 implies that the possibility of learning θ has two effects on the principal's payoff, direct and indirect. The direct effect is that the principal might learn θ in the future and therefore exercise the option at a better time than when the agent recommends to do it. The indirect effect is that the possibility of news arrival also improves communication. Intuitively, since the agent is worried that the principal might learn θ, his benefit from postponing the recommendation to exercise is lower. As a consequence, the agent recommends to exercise the option earlier than in the model without news: $\tilde{X}_A(\theta) < X_A^*(\theta)$. Clearly, both effects increase the value for the principal from keeping authority. Thus, we are able to strengthen the result that centralization leads to better decisions than delegation. Even if the probability of learning θ is infinitely small, the communication equilibrium from Proposition 8 leads to a strictly higher expected payoff to the principal than both unconstrained delegation and constrained delegation for any possible cutoff:

Corollary to Proposition 8. *The principal gets a strictly higher expected payoff in the equilibrium of Proposition 8 than under any interval delegation contract, i.e., when the principal gives authority to the agent to exercise the option subject to constraint $X(t) \leq X'$ for some constant X' (possibly infinite).*

6.2 General distribution

So far, we have assumed that the distribution of types θ is uniform. While this assumption makes the analysis more tractable, it is not critical for the main results. Suppose that type θ is drawn from a continuous distribution Φ with support $\left[\underline{\theta}, \bar{\theta}\right]$, where $0 < \underline{\theta} < \bar{\theta}$, and strictly positive continuous density ϕ. Assume that the distribution satisfies the following assumption:

Assumption 2. *Distribution Φ is such that: (i) $\Phi(\theta) + \frac{b}{I}\theta\phi(\theta)$ is non-decreasing for all $\theta \in [\underline{\theta}, \bar{\theta}]$; (ii) the equation $\mathbb{E}[\tilde{\theta}|\tilde{\theta} \leq \theta] = \frac{I}{I-b}\theta$ has at most one solution on $[\underline{\theta}, \bar{\theta}]$ for $b < 0$.*

The next proposition presents an analog of our main results for a general distribution satisfying Assumption 2.

Proposition 9. *Suppose that Assumption 2 holds and $b \in (-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}I, \frac{\mathbb{E}[\theta]-\underline{\theta}}{\mathbb{E}[\theta]}I)$. For $b \in (-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}I, 0)$, there is a unique solution to $\mathbb{E}[\tilde{\theta}|\tilde{\theta} \leq \theta] = \frac{I}{I-b}\theta$; denote it θ_L. For $b \in (0, \frac{\mathbb{E}[\theta]-\underline{\theta}}{\mathbb{E}[\theta]}I)$, there is at least one solution to $\mathbb{E}[\tilde{\theta}|\tilde{\theta} \geq \theta] = I\frac{\theta}{I-b}$; denote the highest one by θ_H. Then:*

1. *The optimal incentive-compatible threshold schedule $\hat{X}(\theta)$, $\theta \in \Theta$, is given by $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\max\{\theta,\theta_L\}}$ if $b \in (-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}I, 0)$, and $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\min\{\theta,\theta_H\}}$ if $b \in (0, \frac{\mathbb{E}[\theta]-\underline{\theta}}{\mathbb{E}[\theta]}I)$.*
2. *If $b \in (-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}I, 0)$, centralized decision-making with communication implements the optimal mechanism from part 1. Specifically, there exists the following equilibrium, which implements $\hat{X}(\theta)$. The principal's strategy is: (1) to wait if the agent sends message $m = 0$ and to exercise at the first time t at which the agent sends message $m = 1$, provided that $X(t) \in$*

$[X_A^*(\bar{\theta}), X_A^*(\theta_L)]$ *and* $X(t) = \max_{s \le t} X(s)$*; (2) to exercise at the first time* t *at which* $X(t) \ge X_A^*(\theta_L)$*, regardless of the agent's message. The agent's strategy is to send* $m = 0$ *if* $X(t) < X_A^*(\max\{\theta, \theta_L\})$*, and* $m = 1$*, otherwise.*

3. *If* $b \in (0, \frac{\mathbb{E}[\theta]-\underline{\theta}}{\mathbb{E}[\theta]} I)$*, delegation of the decision to the agent when* $X(t)$ *reaches the threshold* $X_A^*(\theta_H)$ *implements the optimal mechanism from part 1. For any* θ*, the agent sends message* $m = 0$ *at any point before he is given authority, and the principal does not exercise the option. If* $\theta \ge \theta_H$*, the agent exercises the option immediately after he is given authority at threshold* $X_A^*(\theta_H)$*, and if* $\theta \le \theta_H$*, the agent exercises the option when* $X(t)$ *first reaches his preferred exercise threshold* $X_A^*(\theta) = \frac{\beta}{\beta-1} \frac{I-b}{\theta}$*.*

The argument behind Proposition 9 is the same as the argument behind the results for the model with uniformly distributed types, but it can be helpful to highlight the sufficient conditions on the distribution. Restriction (i) of Assumption 2 is identical to the restriction in the static delegation problem of Amador and Bagwell (2013) and guarantees that the optimal contract is interval delegation in part 1 of the proposition. Restriction (ii) of Assumption 2 is new. It requires that the agent's optimal exercise threshold for the highest outstanding type $\hat{\theta}$ and the principal's optimal exercise threshold given the belief that $\theta \in [\underline{\theta}, \hat{\theta}]$ cross only once at θ_L (if $b \in (-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]} I, 0)$) or never (if $b < -\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]} I$). This condition implies that in the proof of part 2 of the proposition, it is sufficient to verify the principal's ex-ante IC constraint at first-passage times. A sufficient condition for the two restrictions in Assumption 2 to hold is that density ϕ does not change too rapidly with θ. The uniform distribution case of the basic model with $\underline{\theta} > 0$ is a special case of this result.

7 Conclusion

This paper studies timing decisions in organizations. We consider a problem in which an uninformed principal is deciding when to exercise an option and has to rely on the information of a better-informed but biased agent. We compare the effectiveness of centralized decision-making with communication and the effectiveness of delegation and derive implications for the optimal allocation of authority. Our results emphasize that the equilibrium properties of communication and the optimal allocation of authority for timing decisions crucially depend on whether the agent is biased towards early or late exercise.

We first analyze centralized decision-making, where the principal retains authority and repeatedly communicates with the agent via cheap talk. When the agent favors late exercise, there is often full information revelation but suboptimal delay in option exercise. Moreover, decision-making under centralized decision-making implements the optimal full-commitment mechanism without transfers even though the principal has no commitment power. In contrast, when the agent favors early exercise, there is partial revelation of information, exercise is either unbiased or delayed, and the principal is worse off than under the optimal full-commitment mechanism. The reason for these strikingly different results for the two directions of the agent's bias is the asymmetric nature of time: While the principal can get advice and exercise the option at a later point in time, she cannot go back and exercise the option at an earlier point in time. When the agent is biased towards late exercise, the inability to go back in time creates an implicit commitment device for the principal to follow the agent's recommendation. As a result, centralization features effective communication and implements the optimal full-commitment mechanism. Conversely, when the agent is biased towards early exercise, time does not have built-in commitment, so only partial information revelation is possible.

We next analyze the optimal allocation of authority for timing decisions by studying the principal's choice between centralized decision-making with communication and delegating the decision to the agent. While delegation is always weakly inferior when the agent favors late exercise, it is optimal when the agent favors early exercise and his bias is not very large. If the principal can time the delegation decision strategically, she can implement the second-best by delegating authority at the right time if the agent favors early exercise, but always finds it optimal to retain authority in the case of a late exercise bias.

Our results also imply that in an alternative setting, where the principal is biased towards early exercise (as in the case of an empire-building top manager), it is possible to ensure unbiased decision-making by having an unbiased agent, even if the principal has formal authority.[20] Thus, as in Landier, Sraer, and Thesmar (2009), divergence of preferences between the principal and her subordinate can enhance decision-making quality, although the mechanism in our paper is very different.

[20] For example, Proposition 1 suggests that if $\underline{\theta} = 0$, the principal will exercise the option at the agent's most-preferred threshold, and hence exercise will be unbiased.

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Appendix

A. An example

Here, we present a very simple example that illustrates how communication over time differs from static communication and why the sign of the agent's bias is the first-order determinant of communication efficiency and the allocation of authority.

The principal needs to choose the timing of investment. The agent learns θ at the initial date. It is common knowledge that θ is a draw from a uniform distribution over $[1,2]$. If the principal invests at time t, the principal and the agent obtain the following payoffs at the time of investment:

$$\begin{aligned} U_P(t,\theta) &= C - (t-\theta)^2 \\ U_A(t,\theta) &= C - (t-\theta+b)^2 , \end{aligned}$$

where b is the agent's bias and C is a large enough constant. Time moves continuously starting at zero, and there is no discounting. Thus, given θ, the optimal timing is $t=\theta$ from the position of the principal and $t=\theta-b$ from the position of the agent.

First, suppose that communication occurs only at the initial date $t=0$. In this case, the problem is identical to the quadratic-uniform example in Crawford and Sobel (1982). Consider $b=-\frac{1}{8}$. In addition to the babbling equilibrium, where the agent does not communicate anything and the principal invests at $t=1\frac{1}{2}$, the only equilibrium that exists has two partitions $\left[1,1\frac{3}{4}\right]$ and $\left[1\frac{3}{4},2\right]$. All types in partition $\left[1,1\frac{3}{4}\right]$ send the same message, upon which the principal invests at time $t=1\frac{3}{8}$. Similarly, all types in partition $\left[1\frac{3}{4},2\right]$ send the same message, upon which the principal invests at time $t=1\frac{7}{8}$. Similarly, if $b=\frac{1}{8}$, there is one non-babbling equilibrium, and it consists of two partitions, $\left[1,1\frac{1}{4}\right]$ and $\left[1\frac{1}{4},2\right]$.

Now suppose that the agent and the principal communicate dynamically and $b=-\frac{1}{8}$. In this case, the game has the following equilibrium. The agent of type θ plays a simple threshold strategy of recommending to "wait" ($m=0$) before his most preferred investment time $\theta-b=\theta+\frac{1}{8}$ and recommending to "invest" ($m=1$) once time reaches $\theta+\frac{1}{8}$. Consider the best response of the principal. If the principal receives a recommendation to "invest" at time $t\in\left[1\frac{1}{8},2\frac{1}{8}\right]$, she infers that the agent's type is $t+b=t-\frac{1}{8}$. Since $U_P(t,\theta)$ is strictly decreasing in t in the range $t\geq\theta$, the best response of the principal is to invest immediately upon receiving the recommendation to invest from the agent. If the principal has not received the recommendation to invest from the agent by time t, her optimal strategy is to wait for the agent's recommendation to invest until time τ and to invest at time τ if the agent has not recommended to invest yet, where $\tau\leq 2$ maximizes her expected payoff:

$$C-\int_{t+b}^{\tau+b} b^2 \frac{d\theta}{2-t-b} - \int_{\tau+b}^{2} (\tau-\theta)^2 \frac{d\theta}{2-t-b},$$

yielding $\tau=2+b=1\frac{7}{8}$. That is, the best response of the principal is to follow the agent's recommendation up to $\tau=1\frac{7}{8}$ and to invest then if the agent has not recommended to invest yet. Given that, the agent of type $\theta\leq 1\frac{3}{4}$ does not want to deviate from the strategy of recommending investment at time $\theta+\frac{1}{8}$ because by following this strategy, he gets his preferred investment time. Similarly, no type $\theta>1\frac{3}{4}$ benefits from a deviation, since the principal does not delay investment beyond $\tau=1\frac{7}{8}$. As in the paper, it can be shown that this equilibrium of the cheap talk game implements the optimal commitment mechanism.

Finally, suppose that the agent and the principal communicate dynamically but $b=\frac{1}{8}$, i.e., the agent has a bias for investing earlier than the principal. As above, suppose that the agent of type θ follows the strategy of recommending to "wait" before his most preferred investment time $\theta-b=\theta-\frac{1}{8}$ and recommending to "invest" once time reaches $\theta-\frac{1}{8}$. Now, if the principal receives a recommendation to "invest" at time $t\in\left[\frac{7}{8},1\frac{7}{8}\right]$, she infers that the agent's type is $\theta=t+\frac{1}{8}$ and delays investment by $\frac{1}{8}$ until the time $t+\frac{1}{8}$. Knowing this, the agent deviates from following the above strategy. As a consequence, the equilibrium where the agent credibly communicates his information up to a cutoff does not exist.

This example illustrates two properties of communication over time. First, because the principal cannot go back in time, the set of actions that the principal can take (when to invest) shrinks over time. This gives the principal commitment power not to overrule the agent when the agent is biased towards later investment ($b = -\frac{1}{8}$). As a consequence, communication is very efficient and centralization is preferred to delegation. Second, because the principal can always delay the decision, the commitment power is one-sided: If the agent has a bias for earlier investment ($b = \frac{1}{8}$), the inability to go back in time does not help the principal to commit to follow the agent's recommendation.

B. Proofs

This section of the Appendix contains the main parts of the proofs of Propositions 1-7. The proofs of Lemma 1, Propositions 8 and 9, which describe the extensions of the basic model, and the proofs of some auxiliary results are presented in the Online Appendix.

Proof of Proposition 1. **Part 1:** $b < 0$**.** The proof includes the special case $\underline{\theta} = 0$. First, consider $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}} I$. This implies $b > -I$, and hence $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}} \Leftrightarrow b + I > (I - b)\underline{\theta} \Leftrightarrow \hat{\theta}^* \equiv \frac{I-b}{I+b}\underline{\theta} < 1$. Given that the principal plays the strategy stated in the proposition, it is clear that the strategy of any type θ of the agent is incentive-compatible. Indeed, for any type $\theta \geq \left(\frac{I-b}{I+b}\right)\underline{\theta}$, exercise occurs at his most preferred time. Therefore, no type $\theta \geq \left(\frac{I-b}{I+b}\right)\underline{\theta}$ can benefit from a deviation. Any type $\theta < \left(\frac{I-b}{I+b}\right)\underline{\theta}$ cannot benefit from a deviation either: the agent would lose from inducing the principal to exercise earlier because he is biased towards late exercise, and it is not feasible for him to induce the principal to exercise later because the principal exercises at threshold X^* regardless of the recommendation. For the principal's strategy to be optimal, we need to check that the principal has incentives: to exercise the option immediately when the agent sends message $m = 1$ (the ex-post IC constraint); and not to exercise the option before getting message $m = 1$ (the ex-ante IC constraint). We first show that the principal's ex-post IC constraint is satisfied. If the agent sends a message to exercise when $X(t) < X^*$, the principal learns the agent's type θ and realizes that it is already too late ($X_P^*(\theta) < X_A^*(\theta)$) and thus does not benefit from delaying exercise even further. If the agent sends a message to exercise when $X(t) = X^*$, the principal infers that $\theta \leq \hat{\theta}^*$ and that she will not learn any additional information by waiting more. Given the belief that $\theta \in \left[\underline{\theta}, \hat{\theta}^*\right]$, the optimal exercise threshold for the principal is given by $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+\hat{\theta}^*} = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+\left(\frac{I-b}{I+b}\right)\underline{\theta}} = \frac{\beta}{\beta-1}\frac{I+b}{\underline{\theta}} = X^*$, and hence the ex-post IC constraint is satisfied. Finally, in the Online Appendix, we show that if the principal's ex-ante IC constraint is violated for $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}} I$, then the mechanism derived in Lemma 1 cannot be optimal, which is a contradiction. This completes the proof of existence of equilibrium with continuous exercise for $b > -\frac{1-\underline{\theta}}{1+\underline{\theta}} I$.

Next, consider $b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}} I$. According to Lemma 1, the optimal mechanism is characterized by $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. Clearly, the equilibrium implementing this mechanism exists: The principal exercises at her optimal uninformed threshold $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$ and the agent babbles.

Part 2: $b > 0$**.** According to the proof of Lemma 1, for any $b > 0$, there is a unique exercise policy $\hat{X}(\theta)$ that maximizes the principal's expected utility. Hence, if $b \in (0, \frac{1-\underline{\theta}}{1+\underline{\theta}} I)$, an equilibrium implementing the optimal mechanism can exist only if in this equilibrium exercise happens at $X_A^*(\theta)$ for all $\theta < \frac{I-b}{I+b}$. This, however, is not possible because the principal's optimal exercise time is later than the agent's. Indeed, if a type $\theta < \frac{I-b}{I+b}$ follows the strategy of recommending exercise at his most-preferred threshold $X_A^*(\theta)$, the principal infers the agent's type perfectly and prefers delay over immediate exercise upon getting the recommendation to exercise. Knowing this, the agent is tempted to change his recommendation strategy, mimicking a lower type. Thus, no equilibrium with full separation of types over an interval $\theta < \frac{I-b}{I+b}$ can exist. Finally, if $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}} I$, the equilibrium implementing the optimal mechanism $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$ exists: The principal exercises at her optimal uninformed threshold $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$ and the agent babbles. ■

Proof of Proposition 2. **1. Existence of equilibrium with continuous exercise**. According to the proof of Proposition 1, this equilibrium does not exist for $b > 0$. We next prove that for $b < 0$, this equilibrium exists if and only if $b \geq -I$. Because the agent's IC constraint and the principal's ex-post IC constraint are satisfied, we only need to satisfy the principal's ex-ante IC constraint. Let $V_P^c(X, \hat{\theta})$ denote the principal's expected value in this equilibrium, given that the public state is X and the principal's belief is that θ is uniform over $[0, \hat{\theta}]$. If the agent's type is θ, exercise occurs at threshold $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}(I-b) - I$. Hence,

$$V_P^c(X, \hat{\theta}) = \int_0^{\hat{\theta}} \frac{1}{\hat{\theta}} X^\beta \left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta} \frac{I-\beta b}{\beta-1} d\theta = \frac{(X\hat{\theta})^\beta}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\frac{I-\beta b}{\beta-1}. \tag{10}$$

By stationarity, it is sufficient to verify the principal's ex-ante IC constraint for $\hat{\theta} = 1$, which yields

$$V_P^c(X, 1) \geq \frac{1}{2}X - I \quad \forall X \leq X_A^*(1). \tag{11}$$

We show that (11) is satisfied if and only if $b \geq -I$. Using (10), (11) is equivalent to

$$\frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\frac{I-\beta b}{\beta-1} \geq \max_{X \in (0, X_A^*(1)]} X^{-\beta}\left(\frac{1}{2}X - I\right). \tag{12}$$

The function $X^{-\beta}\left(\frac{1}{2}X - I\right)$ is inverse U-shaped with a maximum at $\bar{X}_u \equiv \frac{\beta}{\beta-1}2I$, where $\bar{X}_u > X_A^*(1) \Leftrightarrow b > -I$. First, suppose that $b < -I$, and hence $\bar{X}_u < X_A^*(1)$. Then, (12) is equivalent to

$$\frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\frac{I-\beta b}{\beta-1} \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right) \Leftrightarrow \frac{1}{\beta+1}(I-b)^{-\beta}(I-\beta b) \geq (2I)^{-\beta} I. \tag{13}$$

Consider $f(b) \equiv (I-b)^{-\beta}(I-\beta b) - (\beta+1)(2I)^{-\beta} I$. Note that $f(-I) = 0$ and $f'(b) > 0$. Hence, $f(b) \geq 0 \Leftrightarrow b \geq -I$, and hence (12) is violated when $b < -I$.

Second, suppose that $b \geq -I$, and hence (13) is satisfied. Since, in this case, $\bar{X}_u \geq X_A^*(1)$, then $\max_{X \in (0, X_A^*(1)]} X^{-\beta}(\frac{1}{2}X - I) \leq \bar{X}_u^{-\beta}(\frac{1}{2}\bar{X}_u - I)$, and hence the inequality (12) follows from the fact that inequality (13) is satisfied. Note also that if $b = -I$, the equilibrium with continuous exercise brings the principal the same payoff as the babbling equilibrium with exercise at $\bar{X}_u$.

2. The necessary and sufficient conditions for a ω-equilibrium to exist. For ω and $\bar{X}$ to constitute an equilibrium, the IC conditions for the principal and the agent must hold. Because the problem is stationary, it is sufficient to only consider the IC conditions for the game up to reaching the first threshold $\bar{X}$. First, consider the agent's problem. Pair $(\omega, \bar{X})$ satisfies the agent's IC condition if and only if types above ω have incentives to recommend exercise ($m = 1$) at threshold $\bar{X}$ rather than to wait, whereas types below ω have incentives to recommend delay ($m = 0$). From the agent's point of view, the set of possible exercise thresholds is given by X: The agent can induce exercise at any threshold in X by sending $m = 1$ at the first instant when $X(t)$ reaches a desired point in X, but cannot induce exercise at any point not in X. This implies that the agent's IC condition holds if and only if type ω is exactly indifferent between exercising the option at threshold $\bar{X}$ and at threshold $\frac{\bar{X}}{\omega}$:

$$\left(\frac{X(t)}{\bar{X}}\right)^\beta \left(\omega\bar{X} + b - I\right) = \left(\frac{X(t)}{\bar{X}/\omega}\right)^\beta \left(\omega\frac{\bar{X}}{\omega} + b - I\right). \tag{14}$$

which simplifies to $\omega\bar{X} + b - I = \omega^\beta\left(\bar{X} + b - I\right)$. Indeed, if (14) holds, then $\left(\frac{X(t)}{\bar{X}}\right)^\beta\left(\theta\bar{X} + b - I\right) \gtreqless$

$\left(\frac{X(t)}{\bar{X}/\omega}\right)^{\beta}\left(\theta\frac{\bar{X}}{\omega}+b-I\right)$ if $\theta \gtrless \omega$. Hence, if type ω is indifferent between exercise at threshold $\bar{X}$ and at threshold $\frac{\bar{X}}{\omega}$, then any higher type strictly prefers recommending exercise at $\bar{X}$, while any lower type strictly prefers recommending delay at $\bar{X}$. By stationarity, if (14) holds, then type ω^2 is indifferent between recommending exercise and recommending delay at threshold $\frac{\bar{X}}{\omega}$, so types in (ω^2, ω) strictly prefer recommending exercise at threshold $\frac{\bar{X}}{\omega}$, and so on. Thus, (14) is necessary and sufficient for the agent's IC condition to hold. Equation (14) is equivalent to (8).

Next, consider the principal's problem. For ω and $\bar{X}$ to constitute an equilibrium, the principal must have incentives: (1) to exercise the option immediately when the agent sends message $m = 1$ at a threshold in X (the ex-post IC constraint) and (2) not to exercise the option before getting message $m = 1$ (the ex-ante IC constraint). Suppose that $X(t)$ reaches threshold $\bar{X}$ for the first time, and the principal receives recommendation $m = 1$ at that instant. By Bayes' rule, the principal updates her beliefs to θ being uniform on $[\omega, 1]$. If the principal exercises immediately, her expected payoff is $\frac{\omega+1}{2}\bar{X} - I$. If the principal delays, she expects that there will be no further informative communication in the continuation game. Thus, upon receiving message $m = 1$ at threshold $\bar{X}$, the principal faces the standard perpetual call option exercise problem (e.g., Dixit and Pindyck, 1994) as if the type of the project were $\frac{\omega+1}{2}$. Immediate exercise is optimal if and only if exercising at threshold $\bar{X}$ dominates waiting until $X(t)$ reaches a higher threshold Z and exercising the option then for any possible $Z > \bar{X}$:

$$\bar{X} \in \arg\max_{Z \geq \bar{X}} \left(\frac{\bar{X}}{Z}\right)^{\beta}\left(\frac{\omega+1}{2}Z - I\right). \tag{15}$$

Using $\bar{X} = Y(\omega)$ and the fact that the right-hand side is an inverted U-shaped function of Z with a maximum at $\frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, the ex-post IC condition for the principal is equivalent to $Y(\omega) \geq \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, i.e., (9). If (9) is violated, the principal delays exercise, so the recommendation loses its responsiveness as the principal does not follow it. In contrast, if (9) holds, the principal's optimal response to getting message $m = 1$ is to exercise immediately. As with the IC condition of the agent, stationarity implies that if (9) holds, then a similar condition holds for all higher thresholds in X. The fact that constraint (9) is an inequality rather than an equality highlights the asymmetric nature of time: When the agent recommends exercise, the principal can either exercise immediately or can delay, but cannot go back in time and exercise in the past, even if it is tempting to do so.

Let $V_P(X(t), \hat{\theta}_t; \omega)$ denote the expected value to the principal in the ω-equilibrium, given that the public state is $X(t)$ and the principal's belief is that θ is uniform over $[0, \hat{\theta}_t]$. In the Online Appendix, we solve for the principal's value in closed form and show that if $\hat{\theta}_t = 1$,

$$V_P(X, 1; \omega) = \frac{1-\omega}{1-\omega^{\beta+1}}\left(\frac{X}{Y(\omega)}\right)^{\beta}\left(\frac{1}{2}(1+\omega)Y(\omega) - I\right) \tag{16}$$

for any $X \leq Y(\omega)$. By stationarity, (16) can be generalized to any $\hat{\theta}$:

$$V_P(X, \hat{\theta}; \omega) = V_P(\hat{\theta}X, 1; \omega) = \frac{1-\omega}{1-\omega^{\beta+1}}\left(\frac{X\hat{\theta}}{Y(\omega)}\right)^{\beta}\left(\frac{1}{2}(1+\omega)Y(\omega) - I\right). \tag{17}$$

The principal's ex-ante IC constraint requires that the principal is better off waiting, rather than exercising immediately, at any time prior to receiving message $m = 1$ at $X(t) \in X$:

$$V_P(X(t), \hat{\theta}_t; \omega) \geq \frac{\hat{\theta}_t}{2}X(t) - I \tag{18}$$

for any $X(t)$ and $\hat{\theta}_t = \sup\{\theta : \bar{X}(\theta) > \max_{s \leq t} X(s)\}$. By stationarity, it is sufficient to verify the ex-ante

IC constraint for $X(t) \leq \bar{X}(1) = Y(\omega)$ and beliefs equal to the prior:

$$V_P(X,1;\omega) \geq \frac{1}{2}X - I \quad \forall X \leq Y(\omega). \tag{19}$$

This inequality states that at any point up to threshold $Y(\omega)$, the principal is better off waiting than exercising the option. If (19) does not hold for some $X \leq Y(\omega)$, then the principal is better off exercising the option when $X(t)$ reaches X, rather than waiting for informative recommendations from the agent. If (19) holds, then the principal does not exercise the option prior to reaching threshold $Y(\omega)$. By stationarity, if (19) holds, then a similar condition holds for the n^{th} partition for any $n \in N$, which implies that (19) and (18) are equivalent.

To summarize, a ω-equilibrium exists if and only if conditions (8), (9), and (19) are satisfied.

3. Existence of ω-equilibria for $b < 0$. We first show that if $b < 0$, then for any positive $\omega < 1$, the principal's ex-post IC is strictly satisfied, i.e., $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$. In the Online Appendix, we prove that $G(\omega) \equiv \frac{(1-\omega^{\beta})(I-b)}{\omega(1-\omega^{\beta-1})} - \frac{\beta}{\beta-1}\frac{2(I-b)}{1+\omega} > 0$ for all $\omega \in [0,1)$, or equivalently, that $Y(\omega) > \frac{\beta}{\beta-1}\frac{2(I-b)}{1+\omega}$. Since $b < 0$, this implies that $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$, and hence the ex-post IC condition of the principal is satisfied for any $\omega < 1$. Thus, the ω-equilibrium exists if and only if the ex-ante IC (19) is satisfied, where $V_P(X,1;\omega)$ is given by (16). Because $X^{-\beta}V_P(X,1;\omega)$ does not depend on X, we can rewrite (19) as

$$X^{-\beta}V_P(X,1;\omega) \geq \max_{X\in(0,Y(\omega)]} X^{-\beta}\left(\frac{1}{2}X - I\right). \tag{20}$$

We pin down the range of ω that satisfies this condition in the following steps, each of which is proved in the Online Appendix.

Step 1: *If $b < 0$, $V_P(X,1;\omega)$ is strictly increasing in ω for any $\omega \in (0,1)$.*

Step 2: $\lim_{\omega\to 1} V_P(X,1;\omega) = V_P^c(X,1)$.

Step 3. *Suppose $-I < b < I$. For ω close enough to zero, the ex-ante IC condition (20) does not hold.*

Step 4. *Suppose $-I < b < I$. Then (20) is satisfied for any $\omega \geq \bar{\omega}$, where $\bar{\omega}$ is the unique solution to $Y(\omega) = \bar{X}_u$. For any $\omega < \bar{\omega}$, (20) is satisfied if and only if $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$.*

Combining the four steps above yields the statement of the proposition for $b < 0$. First, if $b \leq -I$, then $I - b \geq 2I$, and hence $\lim_{\omega\to 1} Y(\omega) = \frac{\beta(I-b)}{\beta-1} \geq \bar{X}_u$. Since $Y(\omega)$ is decreasing, it implies that $Y(\omega) > \bar{X}_u$ for any $\omega < 1$, and hence (20) is equivalent to $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$.According to Steps 1 and 2, for any $\omega < 1$, $V_P(X,1;\omega) < \lim_{\omega\to 1} V_P(X,1;\omega) = V_P^c(X,1)$. As shown in the proof of the equilibrium with continuous exercise above, $X^{-\beta}V_P^c(X,1) \leq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$ for $b \leq -I$, and hence (20) is violated. Hence, there is no ω-equilibrium in this case. Second, if $0 > b > -I$, then according to Step 4, (20) is satisfied for any $\omega \geq \bar{\omega}$, and for any $\omega \leq \bar{\omega}$ (20) is satisfied if and only if $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. The left-hand side of this inequality is increasing in ω according to Step 1, while the right-hand side is constant. Hence, if (20) is satisfied for some $\tilde{\omega}$, it is satisfied for any $\omega \geq \tilde{\omega}$. According to Step 3, for ω close to 0, (20) does not hold. Together, this implies that there exists a unique $\underline{\omega} \in (0,\bar{\omega})$ such that the principal's ex-ante IC (20) holds if and only if $\omega \geq \underline{\omega}$, and that $X^{-\beta}V_P(X,1;\underline{\omega}) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$.

4. Existence of ω-equilibria for $b < 0$.

Since the agent's IC condition is guaranteed by (8), we only have to ensure that the principal's ex-post and ex-ante IC conditions are satisfied. First, we check the principal's ex-post IC condition (9). In the Online Appendix, we prove that (1) $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ has only one solution $\omega = \omega^*$ and (2) $Y(\omega)$ is strictly decreasing in ω for $\omega \in (0,1)$. Since $\lim_{\omega\to 0} Y(\omega) = +\infty$, it follows that the principal's ex-post IC condition is equivalent to $\omega \leq \omega^*$. Next, we check the principal's ex-ante IC condition (19), which is equivalent to (20), where $V_P(X,1;\omega)$ is given by (16). We pin down the range of ω that satisfies this condition in the following steps, which are proved in the Online Appendix.

Step 5: *If $b > 0$, $V_P(X,1;\omega)$ is strictly increasing in ω for any $\omega \in (0,\omega^*)$.*

Step 6: *If $0 < b < I$, then the ex-ante IC condition (20) holds as a strict inequality for $\omega = \omega^*$*.

Combining the steps above yields the statement of the proposition for $b > 0$. Suppose $b < I$. As shown above, the ex-post IC condition holds if and only if $\omega \leq \omega^*$. Recall that $Y(\omega^*) = \frac{\beta}{\beta-1}\frac{2I}{\omega^*+1} < \frac{\beta}{\beta-1}2I = \bar{X}_u$, and hence $\omega^* > \bar{\omega}$. According to Step 4 from the proof of case $b < 0$ above, the ex-ante IC condition (20) is satisfied for any $\omega \geq \bar{\omega}$, and for any $\omega < \bar{\omega}$ (20) is satisfied if and only if $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. The left-hand side of this inequality is increasing in ω for $\omega \leq \omega^*$ according to Step 5, while the right-hand side is constant. Together, this implies that if (20) is satisfied for some $\tilde{\omega}$, it is satisfied for any $\omega \geq \tilde{\omega}$. According to Step 3 from the proof of case $b < 0$ above, for ω close to 0, (20) does not hold. Hence, there exists a unique $\underline{\omega} \in (0,\bar{\omega})$ such that the principal's ex-ante IC (20) holds if and only if $\omega \geq \underline{\omega}$, and $X^{-\beta}V_P(X,1;\underline{\omega}) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. Because, $\underline{\omega} < \bar{\omega}$ and $\bar{\omega} < \omega^*$ by Step 6, we have $\underline{\omega} < \omega^*$. We conclude that both the ex-post and the ex-ante IC conditions hold if and only if $\omega \in [\underline{\omega}, \omega^*]$. Finally, consider $b \geq I$. In this case, all types of agents want immediate exercise, which implies that the principal must exercise the option at the optimal uninformed threshold $\bar{X}_u = \frac{\beta}{\beta-1}2I$. ■

Proof of Proposition 3. 1. Proof for $b < 0$. In the equilibrium with continuous exercise, exercise occurs at the unconstrained optimal time of any type θ of the agent. Therefore, the payoff of any type of the agent is higher in this equilibrium than in any other possible equilibrium. In addition, as Proposition 1 shows, the exercise times implied by the optimal mechanism if the principal could commit to any mechanism, coincide with the exercise times in the equilibrium with continuous exercise. Thus, the principal's expected payoff in this equilibrium exceeds her expected payoff under the exercise rule implied by any other equilibrium.
2. Proof for $b > 0$**.** The expected utility of the principal in the ω-equilibrium is $V_P(X,1;\omega)$, given by (16). As shown in Step 1 of the proof of Proposition 2, $V_P(X,1;\omega)$ is strictly increasing in ω for $\omega \in (0,\omega^*)$. Hence, $V_P(X,1;\omega^*) > V_P(X,1;\omega)$ for any $\omega < \omega^*$. Denote the ex-ante expected utility of the agent (before the agent's type is realized) by $V_A(X,1;\omega)$. Repeating the derivation of the principal's value function $V_P(X,1;\omega)$ in the Online Appendix, it is easy to see that

$$V_A(X,1;\omega) = \frac{1-\omega}{1-\omega^{\beta+1}}\left(\frac{X}{Y(\omega)}\right)^{\beta}\left(\frac{1}{2}(1+\omega)Y(\omega) - (I-b)\right).$$

The only difference of this expression from the expression for $V_P(X,1;\omega)$ given by (16) is that I in the second bracket of (16) is replaced by $(I-b)$. To prove that $V_A(X,1;\omega^*) > V_A(X,1;\omega)$ for any $\omega < \omega^*$, we prove that $V_A(X,1;\omega)$ is strictly increasing in ω for $\omega \in (0,\omega^*)$. The proof repeats the arguments behind Step 1 in the proof of Proposition 2. In particular, we can re-write $V_A(X,1;\omega)$ as $2^{-\beta}X^{\beta}f_1(\omega)\tilde{f}_2(\omega)$, where

$$f_1(\omega) \equiv \frac{(1-\omega)(1+\omega)^{\beta}}{1-\omega^{\beta+1}} \text{ and } \tilde{f}_2(\omega) \equiv \frac{\frac{1}{2}(1+\omega)Y(\omega) - (I-b)}{\left(\frac{1}{2}(1+\omega)Y(\omega)\right)^{\beta}}.$$

As shown in Step 1 in the proof of Proposition 2, $f_1(\omega) > 0$ and $f_1'(\omega) > 0$. In addition, $\tilde{f}_2(\omega) > 0$ because $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega} > \frac{2(I-b)}{1+\omega}$ for any $\omega < \omega^*$, and $\tilde{f}_2'(\omega) > 0$ for the same reasons why $f_2'(\omega) > 0$ in Step 1 in the proof of Proposition 2. Hence, $V_A(X,1;\omega)$ is increasing in $\omega \in (0,\omega^*)$. ■

Proof of Proposition 4. The fact that ω^* decreases in b has been proved in the supplementary analysis for the proof of Proposition 2 in the Online Appendix. We next show that ω^* increases in β. From $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, ω^* solves $F(\omega,\beta) = 0$, where $F(\omega,\beta) = \frac{\beta}{\beta-1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}}\frac{2I}{I-b} - 1 - \frac{1}{\omega}$. Denote the unique solution by $\omega^*(\beta)$. Function $F(\omega,\beta)$ is continuously differentiable in both arguments on $\omega \in (0,1)$, $\beta > 1$. Differentiating $F(\omega^*(\beta),\beta)$ in β, $\frac{\partial\omega^*}{\partial\beta} = -\frac{F_\beta(\omega^*(\beta),\beta)}{F_\omega(\omega^*(\beta),\beta)}$. Since $F(0,\beta) < 0$, $F(1,\beta) = \frac{2b}{I-b} > 0$, and ω^* is the unique solution of $F(\omega,\beta) = 0$ in $(0,1)$, we know that $F_\omega(\omega^*(\beta),\beta) > 0$. In the Online Appendix, we prove that $F_\beta(\omega,\beta) < 0$. Hence, ω^* is strictly increasing in $\beta > 1$. Finally, a standard calculation shows that $\frac{\partial\beta}{\partial\sigma} < 0$, $\frac{\partial\beta}{\partial\alpha} < 0$, and $\frac{\partial\beta}{\partial r} > 0$. Thus, ω^* decreases in β and α and increases in r. ■

Proof of Proposition 5. First, consider the case $b < 0$. Proposition 1 shows that in the dynamic communication game, there exists an equilibrium with continuous exercise, where for each type θ, the option is exercised at threshold $X_A^*(\theta)$. No such equilibrium exists in the static communication game. Indeed, continuous exercise requires that the principal perfectly infers the agent's type. However, since the principal gets this information at time 0, she will exercise the option at $X_P^*(\theta) \neq X_A^*(\theta)$.

We next show that no stationary equilibrium with partitioned exercise exists in the static communication game either. To see this, note that for such an equilibrium to exist, the following conditions must hold. First, the boundary type ω must be indifferent between exercise at $\bar{X}$ and at $\frac{\bar{X}}{\omega}$. Repeating the derivations in Section 4, this requires that (8) holds: $\bar{X} = Y(\omega) \equiv \frac{(1-\omega^\beta)(I-b)}{\omega(1-\omega^{\beta-1})}$ Second, given that the exercise threshold $\bar{X}$ is optimally chosen by the principal given the belief that $\theta \in [\omega, 1]$, it must satisfy $\bar{X} = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$. Combining these two equations, ω must solve equation $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, which can be rewritten as

$$2\beta I\left(\omega - \omega^\beta\right) - (\beta-1)(I-b)(1+\omega)\left(1-\omega^\beta\right) = 0. \quad (21)$$

We next show that the left-hand side of (21) is negative for any $b < 0$ and $\omega < 1$. Since $b < 0$, it is sufficient to prove that

$$2\beta\left(\omega - \omega^\beta\right) < (\beta-1)(1+\omega)\left(1-\omega^\beta\right) \Leftrightarrow s(\omega) \equiv 2\beta\left(\omega - \omega^\beta\right) + (\beta-1)\left(\omega^{\beta+1} - \omega - 1 + \omega^\beta\right) < 0.$$

It is easy to show that $s'(1) = 0$ and that $s''(\omega) < 0 \Leftrightarrow \omega < 1$, and hence $s'(\omega) > 0$ for any $\omega < 1$. Since $s(1) = 0$, then, indeed, $s(\omega) < 0$ for all $\omega < 1$.

Next, consider $b > 0$. As argued above, for ω-equilibrium to exist in the static communication game, ω must satisfy $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$. According to Proposition 2, for $b > 0$, this equation has a unique solution, denoted by ω^*. Thus, among equilibria with $\omega \in [\underline{\omega}, \omega^*]$, which exist in the dynamic communication game, only equilibrium with $\omega = \omega^*$ is an equilibrium of the static communication game. ■

Proof of Proposition 6. Let $VD(X, b)$ denote the expected value to the principal under delegation if the current value of $X(t)$ is X. If the decision is delegated to the agent, exercise occurs at threshold $X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}(I-b) - I$. Hence,

$$VD(X,b) = \int_0^1 X^\beta \left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b) - I\right)d\theta = \frac{X^\beta}{\beta+1}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}\left(\frac{\beta}{\beta-1}(I-b) - I\right).$$

Let $VA(X, b)$ denote the expected value to the principal in the most informative equilibrium of the communication game if the current value of $X(t)$ is X. Using (16) and $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$,

$$VA(X,b) = X^\beta \frac{1-\omega^*(b)}{1-\omega^*(b)^{\beta+1}}\left(\frac{\beta}{\beta-1}\frac{2I}{1+\omega^*(b)}\right)^{-\beta}\frac{I}{\beta-1},$$

where $\omega^*(b)$ is the unique solution to $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$, given b. Because X^β enters as a multiplicative factor in both $VD(X, b)$ and $VA(X, b)$, it is sufficient to compare $VD(b)$ and $VA(b)$, where $VD(b) \equiv X^{-\beta}VD(X, b)$ and $VA(b) \equiv X^{-\beta}VA(X, b)$. First, consider the behavior of $VA(b)$ and $VD(b)$ around $b = I$. According to the supplementary analysis for the proof of Proposition 2 in the Online Appendix, $\lim_{b\to I}\omega^*(b) = 0$. Hence, $\lim_{b\to I} VD(b) = -\infty$ and $\lim_{b\to I} VA(b) = \left(\frac{\beta}{\beta-1}2I\right)^{-\beta}\frac{I}{\beta-1}$. By continuity of $VD(b)$ and $VA(b)$ in b, this implies that there exists $\bar{b} \in (0, I)$, such that for any $b > \bar{b}$, $VA(b) > VD(b)$. In other words, communication dominates delegation if the conflict of interest between the agent and the principal is big enough. Second, consider the behavior of $VA(b)$ and $VD(b)$ for small but positive b. In the Online Appendix, we prove that $\lim_{b\to 0}\frac{VA'(b)}{b} = -\infty$ and $\lim_{b\to 0}\frac{VD'(b)}{b} = -\frac{\beta^2}{\beta^2-1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta-1} > -\infty$. By continuity of $VA'(b)$ and $VD'(b)$ for $b > 0$, there exists $\underline{b} > 0$ such that $VA'(b) < VD'(b)$ for any

$b < \underline{b}$. Because $VA(0) = VD(0)$, $VD(b) - VA(b) = \int_0^b (VD'(y) - VA'(y))\, dy > 0$ for all $b \in (0, \underline{b}]$. Thus, delegation dominates communication if the agent favors early exercise but the bias is low enough. ■

Proof of Proposition 7. Note that the following three inequalities are equivalent: $b \leq \frac{1-\underline{\theta}}{1+\underline{\theta}} I \Leftrightarrow \frac{I-b}{I+b} \geq \underline{\theta} \Leftrightarrow \frac{\beta}{\beta-1}(I+b) \leq \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. Hence, there are two cases. If $b < \frac{1-\underline{\theta}}{1+\underline{\theta}} I$, delegation occurs at threshold $\frac{\beta}{\beta-1}(I+b) = X_A^*\left(\frac{I-b}{I+b}\right)$, where $\frac{I-b}{I+b} > \underline{\theta}$. If $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}} I$, then $\frac{I-b}{I+b} \leq \underline{\theta}$ and delegation occurs at the principal's uninformed exercise threshold $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$.

We prove that neither the agent nor the principal wants to deviate from the specified strategies. First, consider the agent. Given Assumption 1, sending a message $m = 1$ is never beneficial because it does not change the principal's belief and hence her strategy. Hence, the agent cannot induce exercise before he is given authority. After the agent is given authority, his optimal strategy is to: 1) exercise immediately if $b \geq I$, or if $b < I$ and $X_d \geq X_A^*(\theta)$; 2) exercise when $X(t)$ first reaches $X_A^*(\theta)$ if $b < I$ and $X_d < X_A^*(\theta)$. Consider two cases. If $0 < b < \frac{1-\underline{\theta}}{1+\underline{\theta}} I$ $(\leq I)$, then $X_d = X_A^*\left(\frac{I-b}{I+b}\right)$, and hence $X_d < X_A^*(\theta)$ if and only if $\theta < \frac{I-b}{I+b}$. Thus, types below $\frac{I-b}{I+b}$ exercise at $X_A^*(\theta)$ and types above $\frac{I-b}{I+b}$ exercise immediately at X_d, consistent with the equilibrium strategy. Second, if $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}} I$, the agent finds it optimal to exercise immediately at X_d regardless of her type: if $b \geq I$, this is always the case, and if $\frac{1-\underline{\theta}}{1+\underline{\theta}} I \leq b < I$, this is true because $X_A^*(\theta) \leq X_A^*(\underline{\theta}) = \frac{\beta}{\beta-1}\frac{I-b}{\underline{\theta}} \leq \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1} = X_d$. Since $\frac{I-b}{I+b} \leq \underline{\theta}$, this strategy again coincides with the equilibrium strategy. Hence, the agent does not want to deviate.

Next, consider the principal. The above arguments show that the equilibrium exercise times coincide with the exercise times under the optimal mechanism in Lemma 1 for all b. Hence, the principal's expected utility in this equilibrium equals her expected utility in the optimal mechanism. Consider possible deviations of the principal, taking into account that the agent's messages are uninformative and hence the principal does not learn new information by waiting. First, the principal can exercise the option himself, before or after $X(t)$ first reaches X_d. Because a mechanism with such an exercise policy is incentive-compatible, the principal's utility from such a deviation cannot exceed her utility under the optimal mechanism and hence her equilibrium utility. Thus, such a deviation cannot be strictly profitable. Second, the principal can deviate by delegating authority to the agent before or after $X(t)$ first reaches X_d. An agent who receives authority at some point t will exercise immediately if $b \geq I$, or if $b < I$ and $X(t) \geq X_A^*(\theta)$, and will exercise when $X(t)$ first reaches $X_A^*(\theta)$ otherwise. Because a mechanism with such an exercise schedule is incentive-compatible, the principal's utility from this deviation cannot exceed her utility under the optimal mechanism and hence her equilibrium utility. Hence, the principal does not want to deviate either. ■

Proofs of Propositions 8 and 9. See the Online Appendix. ■

Proof of Corollary to Proposition 8. Due to the combination of the direct and the indirect effect, the principal's payoff in the equilibrium of Proposition 8 is strictly higher than her payoff in the equilibrium of Proposition 1, where $\lambda = 0$. Because the equilibrium of Proposition 1 implements the optimal commitment mechanism and this optimal mechanism features interval delegation (unconstrained for $\underline{\theta} = 0$ and constrained up to some cutoff for $\underline{\theta} > 0$), the principal's payoff in the equilibrium of Proposition 1 is weakly higher than her payoff under any interval delegation contract, constrained or unconstrained. ■

For Online Publication: Online Appendix

A. Communication game: Equilibrium notion

This section presents the formal definition of the Perfect Bayesian equilibrium in Markov strategies of the dynamic communication game of Section 4.

This is a dynamic game with observed actions (messages and the exercise decision) and incomplete information (type θ of the agent). Heuristically, the timing of events over an infinitesimal time interval $[t, t+dt]$ prior to option exercise can be described as follows: (1) Nature determines the realization of X_t. (2) The agent sends message $m(t) \in M$ to the principal. (3) The principal decides whether to exercise the option or not. If the option is exercised, the principal obtains the payoff of $\theta X_t - I$, the agent obtains the payoff of $\theta X_t - I + b$, and the game ends. Otherwise, the game continues, and the nature draws $X_{t+dt} = X_t + dX_t$. Because the game ends when the principal exercises the option, we can only consider histories such that the option has not yet been exercised. Then, the history of the game at time t has two components: the sample path of the public state $X(t)$ and the history of messages of the agent: $\mathcal{H}_t = \{X(s), s \leq t; m(s), s < t\}$.

Definition. *Strategies $m^* = \{m_t^*, t \geq 0\}$ and $e^* = \{e_t^*, t \geq 0\}$, beliefs μ^*, and a message space M constitute a **Perfect Bayesian equilibrium in Markov strategies (PBEM)** if:*

1. For every t, $\mathcal{H}_t$, $\theta \in \Theta$, and strategy m,

$$\begin{aligned} & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right) - I + b\right)|\mathcal{H}_t, \theta, \mu^*\left(\cdot|\mathcal{H}_t\right), m^*, e^*\right] \\ \geq \; & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right) - I + b\right)|\mathcal{H}_t, \theta, \mu^*\left(\cdot|\mathcal{H}_t\right), m, e^*\right]. \end{aligned} \tag{22}$$

2. For every t, $\mathcal{H}_t$, $m(t) \in M$, and strategy e,

$$\begin{aligned} & \mathbb{E}\left[e^{-r\tau(e^*)}\left(\theta X\left(\tau\left(e^*\right)\right) - I\right)|\mathcal{H}_t, \mu^*\left(\cdot|\mathcal{H}_t, m(t)\right), m^*, e^*\right] \\ \geq \; & \mathbb{E}\left[e^{-r\tau(e)}\left(\theta X\left(\tau\left(e\right)\right) - I\right)|\mathcal{H}_t, \mu^*\left(\cdot|\mathcal{H}_t, m(t)\right), m^*, e\right]. \end{aligned} \tag{23}$$

3. Bayes' rule is used to update beliefs $\mu^*(\theta|\mathcal{H}_t)$ to $\mu^*(\theta|\mathcal{H}_t, m(t))$ whenever possible: For every $\mathcal{H}_t$ and $m(t) \in M$, if there exists θ such that $m_t^*(\theta, \mathcal{H}_t) = m(t)$, then for all θ

$$\mu^*\left(\theta|\mathcal{H}_t, m(t)\right) = \frac{\mu^*(\theta|\mathcal{H}_t)\mathbf{1}\{m_t^*(\theta, \mathcal{H}_t) = m(t)\}}{\int_{\underline{\theta}}^1 \mu^*(\tilde{\theta}|\mathcal{H}_t)\mathbf{1}\{m_t^*(\tilde{\theta}, \mathcal{H}_t) = m(t)\}d\tilde{\theta}}, \tag{24}$$

where $\mu^*(\theta|\mathcal{H}_0) = \frac{1}{1-\underline{\theta}}$ for $\theta \in \Theta$ and $\mu^*(\theta|\mathcal{H}_0) = 0$ for $\theta \notin \Theta$.

4. For every t, $\mathcal{H}_t$, $\theta \in \Theta$, and $m(t) \in M$,

$$\begin{aligned} m_t^*\left(\theta, \mathcal{H}_t\right) &= m^*\left(\theta, X(t), \mu^*\left(\cdot|\mathcal{H}_t\right)\right); & (25) \\ e_t^*\left(\mathcal{H}_t, m(t)\right) &= e^*\left(X(t), \mu^*\left(\cdot|\mathcal{H}_t, m(t)\right)\right). & (26) \end{aligned}$$

The first three conditions, given by (22)–(24), are requirements of the Perfect Bayesian equilibrium. Inequalities (22) require the equilibrium strategy m^* to be sequentially optimal for the agent for any possible history $\mathcal{H}_t$ and type realization θ. Similarly, inequalities (23) require equilibrium strategy e^* to be sequentially optimal for the principal. Equation (24) requires beliefs to be updated according to Bayes' rule. Finally, conditions (25)–(26) are requirements that the equilibrium strategies and the message space are Markov.

B. Proofs of lemmas and derivations of auxiliary results

Derivation of the optimal exercise policy for the call and put option. Let $V(X)$ be the value of the option to a risk-neutral player if the current value of $X(t)$ is X and the player perfectly knows θ. Because the player is risk-neutral, the expected return from holding the option over a small interval dt, $E\left[\frac{dV}{V}\right]$, must equal the riskless return rdt. By Itô's lemma,

$$dV(X(t)) = \left(V'(X(t))\,\alpha X(t) + \frac{1}{2}V''(X(t))\,\sigma^2 X(t)^2\right)dt + \sigma V'(X(t))\,dW(t),$$

and hence

$$\mathbb{E}\left[\frac{dV(X(t))}{V(X(t))}\right] = rdt \Leftrightarrow \frac{1}{V(X(t))}\left(V'(X(t))\,\alpha X(t) + \frac{1}{2}V''(X(t))\,\sigma^2 X(t)^2\right)dt = rdt,$$

which gives

$$rV_P^*(X,\theta) = \alpha X \frac{\partial V_P^*(X,\theta)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V_P^*(X,\theta)}{\partial X^2}. \tag{27}$$

This is a second-order linear homogeneous ordinary differential equation. The general solution to this equation is $V(X) = D_1 X^{\beta_1} + D_2 X^{\beta_2}$, where D_1 and D_2 are the constants to be determined, and $\beta_1 < 0 < 1 < \beta_2$ are the roots of $\frac{1}{2}\sigma^2\beta(\beta-1) + \alpha\beta - r = 0$. We denote the negative root by $-\delta$ and the positive root by β. To find D_1, D_2, we use two boundary conditions. If exercise of the option occurs at trigger $\bar{X}$ and gives a payoff $p(\bar{X})$, the first boundary condition is $V(\bar{X}) = p(\bar{X})$.

For the call option, the second boundary condition is $\lim_{X\to 0} V(X) = 0$ because zero is an absorbing barrier for the geometric Brownian motion. Hence, $D_1 = 0$. In addition, if θ is known to the principal, then $p_{call}(\bar{X}) = \theta\bar{X} - I$. Hence,

$$V_{call}(X,\bar{X}) = \left(\frac{X}{\bar{X}}\right)^{\beta}(\theta\bar{X} - I). \tag{28}$$

Maximizing $V_{call}(X,\bar{X})$ with respect to $\bar{X}$ to derive the optimal call option exercise policy of the principal gives $\bar{X} = \frac{\beta}{\beta-1}\frac{I}{\theta}$, i.e. (1).

Similarly, for the put option, the second boundary condition is $\lim_{X\to\infty} V(X) = 0$, and hence $D_2 = 0$. Combining it with $V(\bar{X}) = p(\bar{X})$ and using $p_{put}(\bar{X}) = \theta I - \bar{X}$, gives $V_{put}(X,\bar{X}) = \left(\frac{X}{\bar{X}}\right)^{-\delta}(\theta I - \bar{X})$. Maximizing $V_{put}(X,\bar{X})$ with respect to $\bar{X}$ to derive the optimal put option exercise policy of the principal gives $\bar{X} = \frac{\delta}{\delta+1}I\theta$. ■

We next prove two useful auxiliary results, which hold in any threshold-exercise PBEM. The first result shows that in any threshold-exercise PBEM, the option is exercised weakly later if the agent has less favorable information. The second auxiliary result is that it is without loss of generality to reduce the message space significantly. Specifically, for any threshold-exercise equilibrium, there exists an equilibrium with a binary message space $M = \{0, 1\}$ and simple equilibrium strategies that implements the same exercise times and hence features the same payoffs of both players.

Lemma IA.1. *$\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ for any $\theta_1, \theta_2 \in \Theta$ such that $\theta_2 \geq \theta_1$.*

Lemma IA.2. *If there exists a threshold-exercise PBEM with thresholds $\bar{X}(\theta)$, then there exists an equivalent threshold-exercise PBEM with the binary message space $M = \{0, 1\}$ and the following strategies of the agent and the principal and beliefs of the principal:*
1. The agent with type θ sends message $m(t) = 1$ if and only if $X(t) \geq \bar{X}(\theta)$:

$$\bar{m}_t(\theta, X(t), \bar{\mu}(\cdot|\mathcal{H}_t)) = \begin{cases} 1, & \text{if } X(t) \geq \bar{X}(\theta), \\ 0, & \text{otherwise.} \end{cases} \tag{29}$$

2. The posterior belief of the principal at any time t is that θ is distributed uniformly over $[\check{\theta}_t, \hat{\theta}_t]$ for some $\check{\theta}_t$ and $\hat{\theta}_t$ (possibly, equal).
3. The exercise strategy of the principal as a function of the state process and her beliefs is

$$\bar{e}_t(X(t), \check{\theta}_t, \hat{\theta}_t) = \begin{cases} 1, & \text{if } X(t) \geq \check{X}(\check{\theta}_t, \hat{\theta}_t), \\ 0, & \text{otherwise,} \end{cases} \tag{30}$$

for some threshold $\check{X}(\check{\theta}_t, \hat{\theta}_t)$. Function $\check{X}(\check{\theta}_t, \hat{\theta}_t)$ is such that on equilibrium path the option is exercised at the first instant when the agent sends message $m(t) = 1$, i.e., when $X(t)$ hits threshold $\bar{X}(\theta)$ for the first time.

Proof of Lemma IA.1. By contradiction, suppose that $\bar{X}(\theta_1) < \bar{X}(\theta_2)$ for some $\theta_2 > \theta_1$. Using the same arguments as in the derivation of (28) above but for $I - b$ instead of I, it is easy to see that if exercise occurs at a cutoff $\bar{X}$ and the current value of $X(t)$ is $X \leq \bar{X}$, then the agent's expected payoff is given by $\left(\frac{X}{\bar{X}}\right)^{\beta} \left(\theta \bar{X} - I + b\right)$. Hence, because the message strategy of type θ_1 is feasible for type θ_2, the IC condition of type θ_2 implies:

$$\left(\frac{X(t)}{\bar{X}(\theta_2)}\right)^{\beta} \left(\theta_2 \bar{X}(\theta_2) - I + b\right) \geq \left(\frac{X(t)}{\bar{X}(\theta_1)}\right)^{\beta} \left(\theta_2 \bar{X}(\theta_1) - I + b\right). \tag{31}$$

Similarly, because the message strategy of type θ_2 is feasible for type θ_1,

$$\left(\frac{X(t)}{\bar{X}(\theta_1)}\right)^{\beta} \left(\theta_1 \bar{X}(\theta_1) - I + b\right) \geq \left(\frac{X(t)}{\bar{X}(\theta_2)}\right)^{\beta} \left(\theta_1 \bar{X}(\theta_2) - I + b\right). \tag{32}$$

These inequalities imply

$$\begin{aligned} &\theta_2 \bar{X}(\theta_1) \left(1 - \left(\tfrac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta-1}\right) \leq (I - b) \left(1 - \left(\tfrac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta}\right) \\ &\leq \theta_1 \bar{X}(\theta_1) \left(1 - \left(\tfrac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)}\right)^{\beta-1}\right), \end{aligned}$$

which is a contradiction, because $\theta_2 > \theta_1$ and $\frac{\bar{X}(\theta_1)}{\bar{X}(\theta_2)} < 1$. Thus, $\bar{X}(\theta_1) \geq \bar{X}(\theta_2)$ whenever $\theta_2 \geq \theta_1$. ■

Proof of Lemma IA.2. Consider a threshold exercise equilibrium E with an arbitrary message space M^* and equilibrium message strategy m^*, in which exercise occurs at stopping time $\tau^*(\theta) = \inf\{t \geq 0 | X(t) \geq \bar{X}(\theta)\}$ for some set of thresholds $\bar{X}(\theta)$, $\theta \in \Theta$. By Lemma IA.1, $\bar{X}(\theta)$ is weakly decreasing. Define $\theta_l(X) \equiv \inf\{\theta : \bar{X}(\theta) = X\}$ and $\theta_h(X) \equiv \sup\{\theta : \bar{X}(\theta) = X\}$ for any $X \in \mathcal{X}$. We will construct a different equilibrium, denoted by $\bar{E}$, which implements the same equilibrium exercise time $\tau^*(\theta)$ and has the structure specified in the formulation of the lemma. As we will see, it will imply that on the equilibrium path, the principal exercises the option at the first informative time $t \in \mathcal{T}$ at which she receives message $m(t) = 1$, where the set $\mathcal{T}$ of informative times is defined as

$$\mathcal{T} \equiv \{t : X(t) = \bar{X} \text{ for some } \bar{X} \in \mathcal{X} \text{ and } X(s) < \bar{X}\ \forall s < t\},$$

i.e., the set of times when the process $X(t)$ reaches one of the thresholds in X for the first time.

For the collection of strategies (29) and (30) and the corresponding beliefs to be an equilibrium, we need to verify the IC conditions of the agent and the principal.

1 - IC of the agent. The IC condition of the agent requires that any type θ is better off sending a message $m(t) = 1$ when $X(t)$ first reaches $\bar{X}(\theta)$ than following any other strategy. By Assumption 1, a deviation to sending $m(t) = 1$ at any $t \notin \mathcal{T}$ does not lead the principal to change her beliefs, and hence, her behavior. Thus, it is without loss of generality to only consider deviations at $t \in \mathcal{T}$. There are two possible deviations: sending $m(t) = 1$ before $X(t)$ first reaches $\bar{X}(\theta)$ and sending $m(t) = 0$ at that moment and following some other strategy after that. Consider the first deviation: the agent of type θ can send $m(t) = 1$ when $X(t)$ hits threshold $\bar{X}(\hat{\theta})$, $\hat{\theta} > \theta_h(\bar{X}(\theta))$ for the first time, and then the principal will exercise immediately. Consider the second deviation: if type θ deviates to sending $m(t) = 0$ when $X(t)$ hits threshold $\bar{X}(\theta)$, he can then either send $m(t) = 1$ at one of the future $t \in \mathcal{T}$ or continue sending the message $m(t) = 0$ at any future $t \in \mathcal{T}$. First, if the agent deviates to sending $m(t) = 1$ at one of the future $t \in \mathcal{T}$, the principal will exercise the option at one of the thresholds $\hat{Y} \in \mathcal{X}$, $\hat{Y} > \bar{X}(\theta)$. Note that the agent can ensure exercise at any threshold $\hat{Y} \in \mathcal{X}$ such that $\hat{Y} \geq X(t)$ by adopting the equilibrium message strategy of type $\hat{\theta}$ at which $\bar{X}(\hat{\theta}) = \hat{Y}$. Second, if the agent deviates to sending $m(t) = 0$ at all of the future $t \in \mathcal{T}$, there are two cases. If $\bar{X}(\underline{\theta}) = \infty$, the principal will never exercise the option. If $\bar{X}(\underline{\theta}) = \bar{X}_{\max} < \infty$, then the principal's belief when $X(t)$ first reaches $\bar{X}_{\max}$ is that $\theta = \underline{\theta}$, if $\bar{X}(\underline{\theta}) \neq \bar{X}(\theta)$ $\forall \theta \neq \underline{\theta}$, or that $\theta \in [\underline{\theta}, \theta_h(\bar{X}_{\max})]$, otherwise. Upon receiving $m(t) = 0$ at this moment, the principal does not change her belief by Assumption 1 and hence exercises the option at $\bar{X}_{\max} = \bar{X}(\underline{\theta})$. Finally, note that the agent cannot induce exercise at $\hat{Y} \in \mathcal{X}$ if $\hat{Y} < X(t)$: in this case, the principal's belief is that the agent's type is smaller than the type that could induce exercise at $\hat{Y}$ and this belief cannot be reversed according to Assumption 1. Combining all possible deviations, at time t, the agent can deviate to exercise at any $\hat{Y} \in \mathcal{X}$ as long as $\hat{Y} \geq X(t)$. Using the same arguments as in the derivation of (28) above but for $I - b$ instead of I, it is easy to see that the agent's expected utility given exercise at threshold $\bar{X}$ is $\left(\frac{X(t)}{\bar{X}}\right)^{\beta} (\theta\bar{X} - I + b)$. Hence, the IC condition of the agent is that

$$\left(\frac{X(t)}{\bar{X}(\theta)}\right)^{\beta} \left(\theta\bar{X}(\theta) - I + b\right) \geq \max_{\hat{Y} \in \mathcal{X}, \hat{Y} \geq X(t)} \left(\frac{X(t)}{\hat{Y}}\right)^{\beta} \left(\theta\hat{Y} - I + b\right). \quad (33)$$

Let us argue that it holds using the fact that E is an equilibrium. Suppose otherwise. Then, there exists a pair $(\theta, \hat{Y})$ with $\hat{Y} \in \mathcal{X}$ such that

$$\frac{\theta \bar{X}(\theta) - I + b}{\bar{X}(\theta)^{\beta}} < \frac{\theta \hat{Y} - I + b}{\hat{Y}^{\beta}}. \tag{34}$$

However, (34) implies that in equilibrium E type θ is better off deviating from the message strategy $m^*(\theta)$ to the message strategy $m^*\left(\tilde{\theta}\right)$ of type $\tilde{\theta}$, where $\tilde{\theta}$ is any type satisfying $\bar{X}\left(\tilde{\theta}\right) = \hat{Y}$ (since $\hat{Y} \in \mathcal{X}$, at least one such $\tilde{\theta}$ exists). This is impossible, and hence (33) holds. Hence, if the principal plays strategy (30), the agent finds it optimal to play strategy (29).

Given Lemma IA.1 and the fact that the agent plays (29), the posterior belief of the principal at any time t is that θ is distributed uniformly over $[\check{\theta}_t, \hat{\theta}_t]$ for some $\check{\theta}_t$ and $\hat{\theta}_t$ (possibly, equal). Next, consider the IC conditions of the principal. They are comprised of two parts, as evident from (30): we refer to the top line of (30) (exercising immediately when the principal "should" exercise) as the ex-post IC condition, and to the bottom line of (30) (not exercising when the principal "should" wait) as the ex-ante IC condition.

2 - "Ex-post" IC of the principal. First, consider the ex-post IC condition: we prove that the principal exercises immediately if the agent sends message $m(t) = 1$ at the first moment when $X(t)$ hits threshold $\hat{Y}$ for some $\hat{Y} \in \mathcal{X}$ (and sent message $m(t) = 0$ before). Given this message, the principal believes that $\theta \sim Uni[\theta_l(\hat{Y}), \theta_h(\hat{Y})]$. Because the principal expects the agent to play (29), the principal now expects the agent to send $m(t) = 1$ if $X(t) \geq \hat{Y}$, and $m(t) = 0$ otherwise, regardless of $\theta \in [\theta_l(\hat{Y}), \theta_h(\hat{Y})]$. Hence, the principal does not expect to learn any new information. This implies that the principal's problem is now equivalent to the standard option exercise problem with the option paying off $\frac{\theta_l(\hat{Y})+\theta_h(\hat{Y})}{2} X(t)$ upon exercise at time t. Using the same arguments as in the derivation of (28) above, the principal's expected payoff from exercise at threshold $\bar{X}$ is $\left(\frac{X(t)}{\bar{X}}\right)^{\beta}\left(\frac{\theta_l(\hat{Y})+\theta_h(\hat{Y})}{2}\bar{X} - I\right)$, which is an inverse U-shaped function with an unconditional maximum at $\frac{\beta}{\beta-1}\frac{2I}{\theta_l(\hat{Y})+\theta_h(\hat{Y})}$. Thus, the solution of the problem is to exercise the option immediately if and only if

$$X(t) \geq \frac{\beta}{\beta - 1}\frac{2I}{\theta_l(\hat{Y}) + \theta_h(\hat{Y})}. \tag{35}$$

Let us show that any threshold $\hat{Y} \in \mathcal{X}$ and the corresponding type cutoffs $\theta_l(\hat{Y})$ and $\theta_h(\hat{Y})$ in equilibrium E satisfy (35). Consider equilibrium E. For the principal to exercise at threshold $\bar{X}(\theta)$, the value that the principal gets upon exercise must be greater or equal than what she gets from delaying the exercise. The value from immediate exercise equals $\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\bar{X}(\theta) - I$, where $(\mathcal{H}_t, m(t))$ is any history of the sample path of $X(t)$ and equilibrium messages that leads to exercise at time t at threshold $\bar{X}(\theta)$ in equilibrium E. Because waiting until $X(t)$ hits a threshold $\tilde{Y} > \bar{X}(\theta)$ and exercising then is a feasible strategy, the value from delaying exercise is greater or equal than the value from such a deviation, which equals $\left(\frac{\bar{X}(\theta)}{\tilde{Y}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\tilde{Y} - I\right)$. Hence, $\bar{X}(\theta)$ must satisfy

$$\bar{X}(\theta) \in \arg\max_{\tilde{Y} \geq \bar{X}(\theta)} \left(\frac{\bar{X}(\theta)}{\tilde{Y}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m(t)]\tilde{Y} - I\right).$$

Using the fact that the unconditional maximizer of the right-hand side is $\tilde{Y} = \frac{\beta}{\beta-1}\frac{I}{\mathbb{E}[\theta|\mathcal{H}_t, m(t)]}$ and that function $\left(\frac{\bar{X}(\theta)}{\tilde{Y}}\right)^{\beta}\left(\mathbb{E}[\theta|\mathcal{H}_t, m_t]\tilde{Y} - I\right)$ is inverted U-shaped in $\tilde{Y}$, this condition can be equivalently rewritten as

$$\bar{X}(\theta) \geq \frac{\beta}{\beta-1}\frac{I}{\mathbb{E}\left[\theta|\mathcal{H}_t, m(t)\right]},$$

for any history $(\mathcal{H}_t, m(t))$ with $X(t) = \bar{X}(\theta)$ and $m(s) = m_s^*(\mathcal{H}_s, \theta)$ for some $\theta \in [\theta_l(\hat{Y}), \theta_h(\hat{Y})]$ and $s \leq t$. Let $\mathbb{H}_t^*$ denote the set of such histories. Then,

$$\bar{X}(\theta) \geq \frac{\beta}{\beta-1}\max_{(\mathcal{H}_t, m(t))\in\mathbb{H}_t^*}\frac{I}{\mathbb{E}\left[\theta|\mathcal{H}_t, m(t)\right]},$$

or, equivalently,

$$\begin{aligned}
&\tfrac{\beta}{\beta-1}\tfrac{I}{\bar{X}(\theta)} \leq \min\nolimits_{(\mathcal{H}_t, m(t))\in\mathbb{H}_t^*}\mathbb{E}\left[\theta|\mathcal{H}_t, m(t)\right] \\
&\leq \mathbb{E}\left[\mathbb{E}\left[\theta|\mathcal{H}_t, m(t)\right]|\theta \in [\theta_l(\hat{Y}), \theta_h(\hat{Y})], \mathcal{H}_0\right] \\
&= \mathbb{E}\left[\theta|\theta \in [\theta_l(\hat{Y}), \theta_h(\hat{Y})]\right] = \tfrac{\theta_l(\hat{Y})+\theta_h(\hat{Y})}{2},
\end{aligned}$$

where the inequality follows from the fact that the minimum of a random variable cannot exceed its mean, and the first equality follows from the law of iterated expectations. Therefore, when the principal obtains message $m = 1$ at threshold $\hat{Y} \in \mathcal{X}$, her optimal reaction is to exercise immediately. Thus, the ex-post IC condition of the principal is satisfied.

3 - "Ex-ante" IC of the principal. Finally, consider the ex-ante IC condition of the principal stating that the principal is better off waiting following a history $\mathcal{H}_t$ with $m(s) = 0$, $s \leq t$, and $\max_{s\leq t} X(s) < \bar{X}(\underline{\theta})$. Given that the agent follows (30), for any such history $\mathcal{H}_t$, the principal's belief is that $\theta \sim Uni[\underline{\theta}, \theta_l(\hat{Y})]$ for some $\hat{Y} \in \mathcal{X}$. If the principal exercises immediately, her payoff is $\frac{\underline{\theta}+\theta_l(\hat{Y})}{2}X(t) - I$. If the principal waits, her expected payoff is

$$\int_{\underline{\theta}}^{\theta_l(\hat{Y})}\left(\frac{X(t)}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta) - I\right)\frac{1}{\theta_l(\hat{Y}) - \underline{\theta}}d\theta.$$

Suppose that there exists a pair $\hat{Y} \in \mathcal{X}$ and $\tilde{Y} < \lim_{\theta\uparrow\theta_l(\hat{Y})}\bar{X}(\theta)$ such that immediate exercise is optimal when $X(t) = \tilde{Y}$:

$$\frac{\underline{\theta}+\theta_l(\hat{Y})}{2}\tilde{Y} - I > \int_{\underline{\theta}}^{\theta_l(\hat{Y})}\left(\frac{\tilde{Y}}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta) - I\right)\frac{1}{\theta_l(\hat{Y}) - \underline{\theta}}d\theta. \tag{36}$$

We can re-write (36) as

$$\mathbb{E}_\theta\left[\left(\frac{1}{\tilde{Y}}\right)^{\beta}\left(\theta\tilde{Y} - I\right)|\theta < \theta_l(\hat{Y})\right] > \mathbb{E}_\theta\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta) - I\right)|\theta < \theta_l(\hat{Y})\right]. \tag{37}$$

Let us show that if equilibrium E exists, then (37) must be violated. Consider equilibrium E, any type $\tilde{\theta} < \theta_l(\hat{Y})$, time $t < \tau^*\left(\tilde{\theta}\right)$, and any history $(\mathcal{H}_t, m(t))$ such that $X(t) = \tilde{Y}$, $\max_{s\leq t, s\in\mathcal{T}} X(s) = \hat{Y}$, which is consistent with the equilibrium play of type $\tilde{\theta}$, i.e., with $m(s) = m_s^*\left(\tilde{\theta}, \mathcal{H}_s\right)$ $\forall s \leq t$. Let $\mathbb{H}_t^{**}(\tilde{\theta}, \tilde{Y}, \hat{Y})$ denote the set of such histories. Because the principal prefers waiting in equilibrium E, the payoff from immediate exercise in equilibrium E cannot exceed the payoff from waiting:

$$\mathbb{E}\left[\theta\tilde{Y}-I|\mathcal{H}_t,m(t)\right]\leq\mathbb{E}\left[\left(\frac{\tilde{Y}}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)|\mathcal{H}_t,m(t)\right]\Leftrightarrow$$
$$\mathbb{E}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)-\left(\frac{1}{\tilde{Y}}\right)^{\beta}\left(\theta\tilde{Y}-I\right)|\mathcal{H}_t,m(t)\right]\geq 0.$$

This inequality must hold for all histories $(\mathcal{H}_t,m(t))\in\mathbb{H}_t^{**}(\tilde{\theta},\tilde{Y},\hat{Y})$. In any history $(\mathcal{H}_t,m(t))\in\mathbb{H}_t^{**}(\tilde{\theta},\tilde{Y},\hat{Y})$, the option is never exercised by time t if $\tilde{\theta}<\theta_l(\hat{Y})$ and is exercised before time t if $\tilde{\theta}>\theta_l(\hat{Y})$. Therefore, conditional on $\tilde{Y}$, $\hat{Y}$, and $\tilde{\theta}<\theta_l(\hat{Y})$, the distribution of $\tilde{\theta}$ is independent of the sample path of $X(s), s\leq t$. Fixing $\tilde{Y}$ and $\hat{Y}$ and integrating over histories $(\mathcal{H}_t,m(t))\in\mathbb{H}_t^{**}(\tilde{\theta},\tilde{Y},\hat{Y})$ and then over $\tilde{\theta}\in[\underline{\theta},\theta_l(\hat{Y}))$, we obtain that

$$\mathbb{E}_{\tilde{\theta}}\left[\mathbb{E}_{(\mathcal{H}_t,m(t))}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)-\left(\frac{1}{\tilde{Y}}\right)^{\beta}\left(\theta\tilde{Y}-I\right)|\,(\mathcal{H}_t,m(t))\in\mathbb{H}_t^{**}(\tilde{\theta},\tilde{Y},\hat{Y})\right]|\tilde{\theta}\in[\underline{\theta},\theta_l(\hat{Y})),\hat{Y},\tilde{Y}\right]$$
$$\geq 0\Leftrightarrow\mathbb{E}_{\theta}\left[\left(\frac{1}{\bar{X}(\theta)}\right)^{\beta}\left(\theta\bar{X}(\theta)-I\right)-\left(\frac{1}{\tilde{Y}}\right)^{\beta}\left(\theta\tilde{Y}-I\right)|\theta<\theta_l(\hat{Y})\right]\geq 0,$$

where we applied the law of iterated expectations and the conditional independence of the sample path of $X(t)$ and the distribution of $\tilde{\theta}$ (conditional on $\tilde{Y}$, $\hat{Y}$, and $\tilde{\theta}<\theta_l(\hat{Y})$). Therefore, (37) cannot hold. Hence, the ex-ante IC condition of the principal is also satisfied.

Thus, if there exists a threshold exercise equilibrium E where $\tau^*(\theta)=\inf\left\{t\geq 0|X(t)\geq\bar{X}(\theta)\right\}$ for some threshold $\bar{X}(\theta)$, then there exists a threshold exercise equilibrium $\bar{E}$ of the form specified in the lemma, in which the option is exercised at the same time. Finally, let us show that on the equilibrium path, the option is indeed exercised at the first informative time t at which the principal receives message $m(t)=1$. Because any message sent at $t\notin\mathcal{T}$ does not lead to updating of the principal's beliefs and because of the second part of (30), the principal never exercises the option prior to the first informative time $t\in\mathcal{T}$ at which she receives message $m(t)=1$. Consider the first informative time $t\in\mathcal{T}$ at which the principal receives $m(t)=1$. By Bayes' rule, the principal believes that θ is distributed uniformly over $(\theta_l(X(t)),\theta_h(X(t)))$. Equilibrium strategy of the agent (29) implies $X(t)=\bar{X}(\theta)\ \forall\theta\in(\theta_l(X(t)),\theta_h(X(t)))$. Therefore, in equilibrium the principal exercises the option immediately. ■

For Lemma 1, we prove the following lemma, which characterizes the structure of any incentive-compatible decision-making rule and is an analogue of Proposition 1 in Melumad and Shibano (1991) for the payoff specification in our model:

Lemma IA.3. *An incentive-compatible threshold schedule $\hat{X}(\theta)$ must satisfy the following conditions:*

1. *$\hat{X}(\theta)$ is weakly decreasing in θ.*
2. *If $\hat{X}(\theta)$ is strictly decreasing on (θ_1,θ_2), then $\hat{X}(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}$.*

3. *If $\hat{X}(\theta)$ is discontinuous at $\hat{\theta}$, then the discontinuity satisfies*

$$\hat{U}_A\left(\hat{X}^-(\hat{\theta}),\hat{\theta}\right) = \hat{U}_A\left(\hat{X}^+(\hat{\theta}),\hat{\theta}\right), \tag{38}$$

$$\hat{X}(\theta) = \begin{cases} \hat{X}^-(\hat{\theta}), & \forall\theta \in \left[\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})},\hat{\theta}\right), \\ \hat{X}^+(\hat{\theta}), & \forall\theta \in \left(\hat{\theta},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})}\right], \end{cases} \tag{39}$$

$$\hat{X}(\hat{\theta}) \in \left\{\hat{X}^-(\hat{\theta}),\hat{X}^+(\hat{\theta})\right\}, \tag{40}$$

where $\hat{X}^-(\hat{\theta}) \equiv \lim_{\theta\uparrow\hat{\theta}}\hat{X}(\theta)$ and $\hat{X}^+(\hat{\theta}) \equiv \lim_{\theta\downarrow\hat{\theta}}\hat{X}(\theta)$.

Proof of Lemma IA.3. **Proof of Part 1**. The first part of the lemma can be proven by contradiction. Suppose there exist $\theta_1,\theta_2 \in \Theta$, $\theta_2 > \theta_1$, such that $\hat{X}(\theta_2) > \hat{X}(\theta_1)$. Note that $\hat{U}_A(\hat{X},\theta) \equiv X(0)^\beta \hat{X}^{-\beta}(\theta\hat{X} - I + b)$ and $\hat{U}_P(\hat{X},\theta) \equiv X(0)^\beta \hat{X}^{-\beta}(\theta\hat{X} - I)$. The agent's IC constraint for $\theta = \theta_1$ and $\hat{\theta} = \theta_2$, $\hat{U}_A\left(\hat{X}(\theta_1),\theta_1\right) \geq \hat{U}_A\left(\hat{X}(\theta_2),\theta_1\right)$, can be written in the integral form:

$$\int_{\hat{X}(\theta_1)}^{\hat{X}(\theta_2)}\left(\frac{X(0)}{\hat{X}}\right)^\beta \frac{-(\beta-1)\theta_1\hat{X} + \beta(I-b)}{\hat{X}}d\hat{X} \leq 0. \tag{41}$$

Because $\theta_2 > \theta_1$ and $\beta > 1$, (41) implies

$$\int_{\hat{X}(\theta_1)}^{\hat{X}(\theta_2)}\left(\frac{X(0)}{\hat{X}}\right)^\beta \frac{-(\beta-1)\theta_2\hat{X} + \beta(I-b)}{\hat{X}}d\hat{X} < 0,$$

or, equivalently, $\hat{U}_A(\hat{X}(\theta_1),\theta_2) > \hat{U}_A(\hat{X}(\theta_2),\theta_2)$. However, this violates the agent's IC constraint $\hat{U}_A(\hat{X}(\theta_2),\theta_2) \geq \hat{U}_A(\hat{X}(\theta_1),\theta_2)$ for $\theta = \theta_2$ and $\hat{\theta} = \theta_1$. Thus, $\hat{X}(\theta)$ is weakly decreasing in θ.

Proof of Part 2. To prove the second part of the lemma, note that $\hat{U}_A\left(\hat{X},\theta\right)$ is differentiable in θ for all $\hat{X} \in (X(0),\infty)$. Because $\hat{U}_A(\hat{X},\theta)$ is linear in θ, it satisfies the Lipschitz condition and hence is absolutely continuous in θ for all $\hat{X} \in (X(0),\infty)$. Also, $\frac{\partial\hat{U}_A(\hat{X},\theta)}{\partial\theta} = \left(\frac{X(0)}{\hat{X}}\right)^\beta \hat{X}$, and hence $\sup_{\hat{X}\in\mathbf{X}}\left|\frac{\partial\hat{U}_A(\hat{X},\theta)}{\partial\theta}\right|$ is integrable on $\theta \in \Theta$. By the generalized envelope theorem (Milgrom and Segal, 2002), the equilibrium utility of the agent in any mechanism implementing exercise at thresholds $\hat{X}(\theta)$, $\theta \in \Theta$, denoted $V_A(\theta)$, satisfies the integral condition,

$$V_A(\theta) = V_A(\underline{\theta}) + \int_{\underline{\theta}}^{\theta}\left(\frac{X(0)}{\hat{X}(s)}\right)^\beta \hat{X}(s)\,ds.$$

On the other hand, $V_A(\theta) = \hat{U}_A\left(\hat{X}(\theta),\theta\right)$. At any point θ at which $\hat{X}(\theta)$ is strictly decreasing, we have

$$\frac{dV_A(\theta)}{d\theta} = \frac{d\hat{U}_A(\hat{X}(\theta),\theta)}{d\theta} \Leftrightarrow \frac{X(0)^\beta}{\hat{X}(\theta)^\beta}\hat{X}(\theta) = \frac{X(0)^\beta}{\hat{X}(\theta)^\beta}\hat{X}(\theta) - \frac{X(0)^\beta}{\hat{X}(\theta)^\beta}\frac{(\beta-1)\theta\hat{X}(\theta) - \beta(I-b)}{\hat{X}(\theta)}\frac{d\hat{X}(\theta)}{d\theta}.$$

Because $d\hat{X}(\theta) < 0$, it must be that $(\beta-1)\theta\hat{X}(\theta) - \beta(I-b) = 0$. Thus, $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$, which proves the second part of the lemma.

Proof of Part 3. Finally, consider the third part of the lemma. Eq. (38) follows from (39), continuity of $\hat{U}_A(\cdot)$, and incentive compatibility of the mechanism. Otherwise, for example, if $\hat{U}_A\left(\hat{X}^+(\hat{\theta}),\hat{\theta}\right) > \hat{U}_A\left(\hat{X}^-(\hat{\theta}),\hat{\theta}\right)$, then $\hat{U}_A\left(\hat{X}(\hat{\theta}-\varepsilon),\hat{\theta}-\varepsilon\right) = \hat{U}_A\left(\hat{X}^-(\hat{\theta}),\hat{\theta}-\varepsilon\right) < \hat{U}_A\left(\hat{X}^+(\hat{\theta}),\hat{\theta}-\varepsilon\right)$ for a sufficiently small ε, and hence types close enough to $\hat{\theta}$ from below would benefit from a deviation to $\hat{X}^+(\hat{\theta})$, i.e., from mimicking types slightly above $\hat{\theta}$.

Next, we prove (39). First, note that, (39) is satisfied at the boundaries. Indeed, denote $\theta_1^* \equiv \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})}$ and suppose that $\hat{X}(\theta_1^*) \neq \hat{X}^-(\hat{\theta})$. Then, by the first part of the lemma, $\hat{X}(\theta_1^*) > \hat{X}^-(\hat{\theta})$. Because $\hat{X}^-(\hat{\theta}) \equiv \lim_{\theta\uparrow\hat{\theta}} \hat{X}(\theta)$, there exists $\varepsilon > 0$ such that $\hat{X}(\theta_1^*) > \hat{X}(\hat{\theta}-\varepsilon) \geq \hat{X}^-(\hat{\theta})$. Because the function $\hat{U}_A(x,\theta_1^*)$ has a maximum at $\hat{X}^-(\hat{\theta})$ and is strictly decreasing for $x > \hat{X}^-(\hat{\theta})$, this would imply $\hat{U}_A(\hat{X}(\theta_1^*),\theta_1^*) < \hat{U}_A(\hat{X}\left(\hat{\theta}-\varepsilon\right),\theta_1^*)$, and hence would contradict the IC condition for type θ_1^*. The proof for the boundary $\theta_2^* \equiv \frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})}$ is similar.

We next prove (39) for interior values of θ. First, suppose that $\hat{X}(\theta) \neq \hat{X}^-(\hat{\theta})$ for some $\theta \in \left(\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})},\hat{\theta}\right)$. By part 1 of the lemma, $\hat{X}(\theta) > \hat{X}^-(\hat{\theta})$. By IC, $\hat{U}_A(\hat{X}(\theta),\theta) \geq \hat{U}_A(\hat{X}^-(\hat{\theta}),\theta)$, which can be written in the integral form as:

$$\int_{\hat{X}^-(\hat{\theta})}^{\hat{X}(\theta)} \left(\frac{X(0)}{\xi}\right)^\beta \frac{-(\beta-1)\theta\xi+\beta(I-b)}{\xi} d\xi \geq 0.$$

The function under the integral on the left-hand side is strictly decreasing in θ and the interval $(\hat{X}^-(\hat{\theta}),\hat{X}(\theta))$ is non-empty. Thus, we can replace θ by $\tilde{\theta} < \theta$ under the integral and get a strict inequality: $\hat{U}_A(\hat{X}(\theta),\tilde{\theta}) > \hat{U}_A(\hat{X}^-(\hat{\theta}),\tilde{\theta})$ for every $\tilde{\theta} \in [\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})},\theta)$. However, this contradicts $\hat{X}^-(\hat{\theta}) = \arg\max_x \hat{U}_A(x,\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})})$. Second, suppose that $\hat{X}(\theta) \neq \hat{X}^+(\hat{\theta})$ for some $\theta \in \left(\hat{\theta},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})}\right)$. By part 1 of the lemma, $\hat{X}(\theta) < \hat{X}^+(\hat{\theta})$. By incentive compatibility, $\hat{U}_A\left(\hat{X}(\theta),\theta\right) \geq \hat{U}_A\left(\hat{X}^+(\theta),\theta\right)$, which can be written as

$$\int_{\hat{X}(\theta)}^{\hat{X}^+(\theta)} \left(\frac{X(0)}{\xi}\right)^\beta \frac{-(\beta-1)\theta\xi+\beta(I-b)}{\xi} d\xi \leq 0.$$

The function under the integral on the left-hand side is strictly decreasing in θ and the interval $(\hat{X}(\theta),\hat{X}^+(\hat{\theta}))$ is non-empty. Therefore, we can replace θ by $\tilde{\theta} > \theta$ under the integral and get a strict inequality, $\hat{U}_A\left(\hat{X}(\theta),\tilde{\theta}\right) > \hat{U}_A\left(\hat{X}^+(\theta),\tilde{\theta}\right)$, for every $\tilde{\theta} \in \left(\theta,\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})}\right]$. However, this contradicts $\hat{X}^+(\hat{\theta}) = \arg\max_x \hat{U}_A\left(x,\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})}\right)$.

Finally, (40) follows from the continuity of $\hat{U}_A(\cdot)$ and incentive compatibility of $\hat{X}(\theta)$. Because $\hat{\theta} \in (\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^-(\hat{\theta})},\frac{\beta}{\beta-1}\frac{I-b}{\hat{X}^+(\hat{\theta})})$, every policy with thresholds strictly below $\hat{X}^-(\hat{\theta})$ or strictly above $\hat{X}^+(\hat{\theta})$ is strictly dominated by $\hat{X}^-(\hat{\theta})$ and $\hat{X}^+(\hat{\theta})$, respectively, and thus cannot be incentive-compatible. Suppose that $\hat{X}(\hat{\theta}) \in \left(\hat{X}^-(\hat{\theta}),\hat{X}^+(\hat{\theta})\right)$. Incentive compatibility and (38) imply $\hat{U}_A\left(\hat{X}(\hat{\theta}),\hat{\theta}\right) \geq \hat{U}_A\left(\hat{X}^-(\hat{\theta}),\hat{\theta}\right) = \hat{U}_A\left(\hat{X}^+(\hat{\theta}),\hat{\theta}\right)$. Because $\hat{U}_A\left(x,\hat{\theta}\right)$ is strictly increasing in x for $x < \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$ and strictly decreasing in x for $x > \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$, the inequality must be strict: $\hat{U}_A\left(\hat{X}(\hat{\theta}),\hat{\theta}\right) > \hat{U}_A\left(\hat{X}^-(\hat{\theta}),\hat{\theta}\right) = \hat{U}_A\left(\hat{X}^+(\hat{\theta}),\hat{\theta}\right)$. However, this together with (39) and continuity of $\hat{U}_A(\cdot)$ implies that types close enough to $\hat{\theta}$ benefit from a deviation to threshold $\hat{X}(\hat{\theta})$. Hence, it must be that $\hat{X}(\hat{\theta}) \in \left\{\hat{X}^-(\hat{\theta}),\hat{X}^+(\hat{\theta})\right\}$. ■

Proof of Lemma 1. We show that for all parameter values, except the case $b = -I$ and $\underline{\theta} = 0$, there exists a unique optimal contract, and it takes the form specified in the lemma. When $b = -I$ and $\underline{\theta} = 0$, the optimal contract is not unique, but the flat contract specified in the lemma is optimal. To prove the lemma, we consider three cases: $b \geq I$, $b \in [-I, I)$, and $b < -I$. Denote the flat contract from the first part of the lemma by $\hat{X}_{flat}(\theta)$, the contract from the second part by $\hat{X}_{-}(\theta)$, and the contract from the third part by $\hat{X}_{+}(\theta)$.

Case 1: $b \geq I$. In this case, all types of agents want to exercise the option immediately. This means that any incentive-compatible contract must be flat. Among flat contracts, the one that maximizes the payoff to the principal solves

$$\arg\max_{x} \int_{\underline{\theta}}^{1} \frac{\theta x - I}{x^{\beta}} d\theta = \frac{2\beta}{\beta - 1} \frac{I}{1 + \underline{\theta}}. \tag{42}$$

Case 2: $b \in [-I, I)$. The proof for this case proceeds in two steps. First, we show that the optimal contract cannot have discontinuities, except the case $b = -I$. Second, we show that the optimal continuous contract is as specified in the lemma.

Step 1: *If $b > -I$, the optimal contract is continuous.* Indeed, by contradiction, suppose that the optimal contract $C = \left\{\hat{X}(\theta), \theta \in \Theta\right\}$ has a discontinuity at some point $\hat{\theta} \in (\underline{\theta}, 1)$. By Lemma IA.3, the discontinuity must satisfy (38)–(40). In particular, (39) implies that there exist $\theta_1 < \hat{\theta}$ and $\theta_2 > \hat{\theta}$ such that $\hat{X}(\theta) = X_A^*(\theta_1)$ for $\theta \in \left[\theta_1, \hat{\theta}\right)$ and $\hat{X}(\theta) = X_A^*(\theta_2)$ for $\theta_2 \in \left(\hat{\theta}, \theta_2\right]$. For any $\tilde{\theta}_2 \in (\hat{\theta}, \theta_2]$, consider a contract $C_1 = \left\{\hat{X}_1(\theta), \theta \in \Theta\right\}$, defined as

$$\hat{X}_1(\theta) = \begin{cases} \hat{X}(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_1] \cup [\theta_2, 1], \\ X_A^*(\theta_1), & \text{if } \theta \in \left[\theta_1, \tilde{\theta}\right), \\ X_A^*\left(\tilde{\theta}_2\right), & \text{if } \theta \in \left(\tilde{\theta}, \tilde{\theta}_2\right], \\ X_A^*(\theta), & \text{if } \theta \in \left(\tilde{\theta}_2, \theta_2\right), \end{cases}$$

where $\tilde{\theta} = \tilde{\theta}\left(\tilde{\theta}_2\right)$ satisfies

$$\frac{\tilde{\theta} X_A^*(\theta_1) - I + b}{X_A^*(\theta_1)^{\beta}} = \frac{\tilde{\theta} X_A^*\left(\tilde{\theta}_2\right) - I + b}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}}. \tag{43}$$

Because $X^{-\beta}(\theta X - I + b)$ is maximized at $X_A^*(\theta)$, the function $\pi(\theta) \equiv \frac{\theta X_A^*(\theta_1) - I + b}{X_A^*(\theta_1)^{\beta}} - \frac{\theta X_A^*\left(\tilde{\theta}_2\right) - I + b}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}}$ satisfies $\pi(\theta_1) > 0 > \pi\left(\tilde{\theta}_2\right)$, and hence, by continuity of $\pi(\theta)$, there exists $\tilde{\theta} \in \left(\theta_1, \tilde{\theta}_2\right)$ such that $\pi\left(\tilde{\theta}\right) = 0$, i.e., (43) is satisfied. Intuitively, contract C_1 is the same as contract C, except that it substitutes a subset $\left[\tilde{\theta}_2, \theta_2\right]$ of the flat region with a continuous region where $\hat{X}_1(\theta) = \frac{\beta}{\beta - 1} \frac{I - b}{\theta}$. Because contract C is incentive-compatible and $\tilde{\theta}$ satisfies (43), contract C_1 is incentive-compatible too. Below we show that the payoff to the principal from contract C_1 exceeds the payoff to the principal from contract C for $\tilde{\theta}_2$ very close to θ_2. Because $\hat{X}_1(\theta) = \hat{X}(\theta)$ for $\theta \leq \theta_1$ and $\theta \geq \theta_2$, it is enough to restrict attention to the payoff in the range $\theta \in (\theta_1, \theta_2)$. The payoff to the principal from contract C_1 in this range, divided by $X(0)^{\beta} \frac{1}{1 - \underline{\theta}}$, is

$$\int_{\theta_1}^{\tilde{\theta}(\tilde{\theta}_2)} \frac{\theta X_A^*(\theta_1) - I}{X_A^*(\theta_1)^{\beta}} d\theta + \int_{\tilde{\theta}(\tilde{\theta}_2)}^{\tilde{\theta}_2} \frac{\theta X_A^*\left(\tilde{\theta}_2\right) - I}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}} d\theta + \int_{\tilde{\theta}_2}^{\theta_2} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^{\beta}} d\theta. \tag{44}$$

The derivative of (44) with respect to $\tilde{\theta}_2$, after the application of (43) and Leibniz's integral rule, is

$$\int_{\tilde{\theta}}^{\tilde{\theta}_2} \frac{\beta I-(\beta-1)\,\theta X_A^*\left(\tilde{\theta}_2\right)}{X_A^*\left(\tilde{\theta}_2\right)^{\beta+1}} X_A^{*\prime}\left(\tilde{\theta}_2\right) d\theta + b\left(\frac{1}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}}-\frac{1}{X_A^*\left(\theta_1\right)^{\beta}}\right)\frac{d\tilde{\theta}}{d\tilde{\theta}_2}. \tag{45}$$

Because $X_A^{*\prime}(\theta) = -\frac{X_A^*(\theta)}{\theta}$, the first term of (45) can be simplified to

$$\frac{(\beta-1)\,X_A^*\left(\tilde{\theta}_2\right)\frac{\tilde{\theta}_2^2-\tilde{\theta}^2}{2}-\beta I\left(\tilde{\theta}_2-\tilde{\theta}\right)}{X_A^*\left(\tilde{\theta}_2\right)^{\beta+1}}\frac{X_A^*\left(\tilde{\theta}_2\right)}{\tilde{\theta}_2} = \beta\frac{\tilde{\theta}_2-\tilde{\theta}}{\tilde{\theta}_2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}\left[\frac{I-b}{\tilde{\theta}_2}\frac{\tilde{\theta}_2+\tilde{\theta}}{2}-I\right]. \tag{46}$$

From (43), $\frac{d\tilde{\theta}}{d\tilde{\theta}_2} = (\frac{\theta_1^{\beta}}{\theta_1}-\frac{\tilde{\theta}_2^{\beta}}{\tilde{\theta}_2})^{-1}(\beta-1)\,\tilde{\theta}_2^{\beta-2}(\tilde{\theta}-\tilde{\theta}_2)$. Using this and (43), the second term of (45) can be simplified to

$$\frac{b}{X_A^*\left(\tilde{\theta}_2\right)^{\beta}}\left(1-\left(\frac{\tilde{\theta}_2}{\theta_1}\right)^{-\beta}\right)\frac{d\tilde{\theta}}{d\tilde{\theta}_2} = \beta\frac{\tilde{\theta}_2-\tilde{\theta}}{\tilde{\theta}_2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}\left(\frac{\tilde{\theta}}{\tilde{\theta}_2}\right)b. \tag{47}$$

Adding (46) and (47), the derivative of the principal's payoff with respect to $\tilde{\theta}_2$ is $-\beta\frac{(\tilde{\theta}_2-\tilde{\theta})^2}{2\tilde{\theta}_2^2}X_A^*\left(\tilde{\theta}_2\right)^{-\beta}(I+b)$, which is strictly negative for any $b > -I$. By the mean value theorem, if $U_P\left(\tilde{\theta}_2\right)$ stands for the expected principal's utility from contract C, then $\frac{U_P\left(\tilde{\theta}_2\right)-U_P(\theta_2)}{\tilde{\theta}_2-\theta_2} = U_P'(\hat{\theta}_2) < 0$ for some $\hat{\theta}_2 \in \left(\tilde{\theta}_2, \theta_2\right)$, and hence a deviation from contract C to contract C_1 is beneficial for the principal. Hence, contract C cannot be optimal for $b > -I$.

Next, suppose $b = -I$. In this case, the derivative of (44) with respect to $\tilde{\theta}_2$ is zero for any $\tilde{\theta}_2 \in (\hat{\theta}, \theta_2]$. It can be similarly shown that if, instead, we replace a subset $\left[\theta_1, \tilde{\theta}_1\right]$ of the flat region $[\theta_1, \theta_2]$ with a continuous region where $\hat{X}_1(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$, then the derivative of the principal's utility with respect to $\tilde{\theta}_1$ is zero for any $\tilde{\theta}_1 \in [\theta_1, \hat{\theta})$. Combining the two arguments, contract C gives the principal the same expected utility as the contract where the flat region $[\theta_1, \theta_2]$ is replaced by a continuous region with $\hat{X}_1(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the rest of the contract is unchanged. Thus, if a discontinuous contract is optimal, then there exists an equivalent continuous contract, which contains a strictly decreasing region and which is also optimal.

Step 2: *Optimal continuous contract.* We prove that among continuous contracts satisfying Lemma IA.3, the one specified in Lemma 1 maximizes the payoff to the principal. By Lemma IA.3 and continuity of the contract, it is sufficient to restrict attention to contracts that are combinations of, at most, one downward sloping part $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ and two flat parts: any contract that has at least two disjoint regions with $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ will exhibit discontinuity. Consider a contract such that $\hat{X}(\theta)$ is flat for $\theta \in [\underline{\theta}, \theta_1]$, is downward-sloping with $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$ for $\theta \in [\theta_1, \theta_2]$, and is again flat for $\theta \in [\theta_2, 1]$, for some $\theta_1 \in [0, \theta_2]$ and $\theta_2 \in [\theta_1, 1]$. This consideration allows for all possible cases, because it can be that $\theta_1 = \underline{\theta}$ and/or $\theta_2 = 1$, or $\theta_1 = \theta_2$). The payoff to the principal, divided by $X(0)^{\beta}\frac{1}{1-\underline{\theta}}$, is

$$P = \int_{\underline{\theta}}^{\theta_1}\frac{\theta X_A^*(\theta_1)-I}{X_A^*(\theta_1)^{\beta}}d\theta + \int_{\theta_1}^{\theta_2}\frac{\theta X_A^*(\theta)-I}{X_A^*(\theta)^{\beta}}d\theta + \int_{\theta_2}^{1}\frac{\theta X_A^*(\theta_2)-I}{X_A^*(\theta_2)^{\beta}}d\theta. \tag{48}$$

Since $X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$, the derivative with respect to θ_1 is

$$\frac{\partial P}{\partial \theta_1} = \int_{\underline{\theta}}^{\theta_1} \frac{\beta I - (\beta-1)\theta X_A^*(\theta_1)}{X_A^*(\theta_1)^{\beta+1}} X_A^{*\prime}(\theta_1)\, d\theta = -\frac{\beta}{X_A^*(\theta_1)^{\beta}}\left[\frac{I+b}{2} - I\frac{\underline{\theta}}{\theta_1} + \left(\frac{\underline{\theta}}{\theta_1}\right)^2 \frac{I-b}{2}\right].$$

First, suppose $\underline{\theta} > 0$. Then $x = \frac{\underline{\theta}}{\theta_1}$ takes values between $\frac{\underline{\theta}}{\theta_2}$ and 1. Since $b \in [-I, I)$, the function $x^2\frac{I-b}{2} - Ix + \frac{I+b}{2}$ is U-shaped and has two roots, 1 and $\frac{I+b}{I-b}$, which coincide for $b = 0$. If $b \in [0, I)$, this function is strictly positive for $x < 1$ because $\frac{I+b}{I-b} \geq 1$. Hence, $\frac{\partial P}{\partial \theta_1} < 0$ for $\theta_1 > \underline{\theta}$, which implies that (48) is maximized at $\theta_1 = \underline{\theta}$. If $-I < b < 0$, then $0 < \frac{I+b}{I-b} < 1$ and hence $\frac{\partial P}{\partial \theta_1} < 0$ when $\frac{\underline{\theta}}{\theta_1} < \frac{I+b}{I-b}$ or $\frac{\underline{\theta}}{\theta_1} > 1$, and $\frac{\partial P}{\partial \theta_1} > 0$ when $\frac{\underline{\theta}}{\theta_1} \in \left(\frac{I+b}{I-b}, 1\right)$. Because $\frac{\underline{\theta}}{\theta_1} \leq 1$, we conclude that (48) is increasing in θ_1 in the range $\theta_1 < \frac{I-b}{I+b}\underline{\theta}$ and decreasing in θ_1 in the range $\theta_1 > \frac{I-b}{I+b}\underline{\theta}$. Therefore, if $-I < b < 0$, (48) reaches its maximum at $\theta_1 = \min\left\{\frac{I-b}{I+b}\underline{\theta}, 1\right\}$. In particular, the maximum is achieved at $\theta_1 = \frac{I-b}{I+b}\underline{\theta}$ if $b \in [-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0)$, and $\theta_1 = \theta_2$ if $-I < b \leq -\frac{1-\underline{\theta}}{1+\underline{\theta}}I$. Finally, if $b = -I$, then $\frac{I+b}{I-b} = 0$ and hence $\frac{\partial P}{\partial \theta_1} > 0$, i.e., (48) is increasing in θ_1. Thus, (48) is also maximized at $\theta_1 = \theta_2$.

Next, suppose $\underline{\theta} = 0$. Then $\frac{\partial P}{\partial \theta_1} < 0$ if $-I < b < I$ and $\frac{\partial P}{\partial \theta_1} = 0$, otherwise. Hence, for $-I < b < I$ and $\underline{\theta} = 0$, (48) is maximized at $\theta_1 = 0 = \underline{\theta}$. If $b = -I$ and $\underline{\theta} = 0$, the principal's utility does not depend on θ_1.

Next, the derivative of (48) with respect to θ_2 is

$$\frac{\partial P}{\partial \theta_2} = \int_{\theta_2}^{1} \frac{\beta I - (\beta-1)\theta X_A^*(\theta_2)}{X_A^*(\theta_2)^{\beta+1}} X_A^{*\prime}(\theta_2)\, d\theta = \frac{\beta(1-\theta_2)}{2\theta_2^2 X_A^*(\theta_2)^{\beta}}\left(I - b - (I+b)\theta_2\right). \tag{49}$$

1) If $b \in [-I, 0)$, then $I - b - (I+b)\theta_2 \geq I - b - (I+b) > 0$, and hence (49) is positive for any $\theta_2 \in [\underline{\theta}, 1)$. Therefore, (48) is maximized at $\theta_2 = 1$. Combining this with the conclusions for θ_1 above, we get:

1a) For $\underline{\theta} > 0$: If $b \in [-\frac{1-\underline{\theta}}{1+\underline{\theta}}I, 0]$, then $\theta_1^* = \frac{I-b}{I+b}\underline{\theta}$ and $\theta_2^* = 1$, which together with continuity of the contract gives $\hat{X}_-(\theta)$. If $b \in [-I, -\frac{1-\underline{\theta}}{1+\underline{\theta}}I]$, then $\theta_1^* = \theta_2$ and $\theta_2^* = 1$, i.e., the optimal contract is flat. As shown above, among flat contracts, the one that maximizes the principal's payoff is $\hat{X}_{flat}(\theta)$. Note that this result implies that the optimal contract is unique among both continuous and discontinuous contracts even if $b = -I$. Indeed, Step 1 shows that the principal's utility in any discontinuous contract is the same as in a continuous contract with a strictly decreasing region. Because the optimal contract among continuous contracts is unique and is strictly flat, the principal's utility in any discontinuous contract is strictly smaller than in the flat contract, which proves uniqueness.

1b) For $\underline{\theta} = 0$: If $b \in (-I, 0)$, then $\theta_1^* = 0$ and $\theta_2^* = 1$, i.e., the optimal contract is $X_A^*(\theta)$ for all θ, consistent with $\hat{X}_-(\theta)$. If $b = -I$, then $\theta_2^* = 1$ and $\theta_1^* \in [0, 1]$, i.e., multiple optimal contracts exist (including some discontinuous contracts, as shown before). The flat contract given by $\hat{X}_{flat}(\theta)$ is one of the optimal contracts.

2) If $b \in [0, I)$, we have shown that $\theta_1^* = \underline{\theta}$ for any $\underline{\theta} \geq 0$, and hence we need to choose $\theta_2 \in [\underline{\theta}, 1]$. According to (49), $\frac{\partial P}{\partial \theta_2} > 0$ for $\theta_2 < \frac{I-b}{I+b}$ and $\frac{\partial P}{\partial \theta_2} < 0$ for $\theta_2 > \frac{I-b}{I+b}$. Since $b \geq 0$, $\frac{I-b}{I+b} < 1$. Also, $\frac{I-b}{I+b} \geq \underline{\theta} \Leftrightarrow b \leq \frac{1-\underline{\theta}}{1+\underline{\theta}}I$. Hence, if $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}}I$, then $\frac{\partial P}{\partial \theta_2} < 0$ for $\theta_2 > \underline{\theta}$, and hence (48) is maximized at $\theta_2 = \underline{\theta}$. Thus, for $b \geq \frac{1-\underline{\theta}}{1+\underline{\theta}}I$, the optimal contract is flat, which gives $\hat{X}_{flat}(\theta)$. Finally, if $b \in (0, \frac{1-\underline{\theta}}{1+\underline{\theta}}I]$, then (48) is increasing in θ_2 up to $\frac{I-b}{I+b}$ and decreasing after that. Hence, (48) is maximized at $\theta_2 = \frac{I-b}{I+b}$. Combined with $\theta_1 = \underline{\theta}$ and continuity of the contract, this gives $\hat{X}_+(\theta)$.

Case 3: $b < -I$. We show that the optimal contract is flat with $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+1}$. The proof proceeds in three steps. First, we show that the optimal contract cannot have any strictly decreasing regions and hence can only consist of flat regions. Second, we show that any contract with two flat regions is strictly dominated by a completely flat contract. Third, we show that any contract with at least three flat regions cannot be optimal. Combined, these steps imply that the optimal contract can only have one flat region, i.e., is completely flat. Combining this with (42) gives $\hat{X}_{flat}(\theta)$ and completes the proof of this case.

Step 1: *The optimal contract cannot have any strictly decreasing regions.*

Consider a contract with a strictly decreasing region. According to Lemma IA.3, any strictly decreasing region is characterized by $\hat{X}(\theta) = X_A^*(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\theta}$. Consider θ_1 and θ_2 such that $\hat{X}(\theta) = X_A^*(\theta)$ for $\theta \in [\theta_1, \theta_2]$. For any $\hat{\theta}_2 \in (\theta_1, \theta_2)$, consider a contract $C_2 = \left\{\hat{X}_2(\theta), \theta \in \Theta\right\}$, defined as

$$\hat{X}_2(\theta) = \begin{cases} \hat{X}(\theta), & \text{if } \theta \in [\underline{\theta}, \theta_1] \cup [\theta_2, 1], \\ X_A^*(\theta), & \text{if } \theta \in \left[\theta_1, \hat{\theta}_2\right), \\ X_A^*(\hat{\theta}_2), & \text{if } \theta \in \left(\hat{\theta}_2, \hat{\theta}\right], \\ X_A^*(\theta_2), & \text{if } \theta \in \left(\hat{\theta}, \theta_2\right), \end{cases}$$

where $\hat{\theta} = \hat{\theta}(\hat{\theta}_2)$ satisfies

$$\frac{\hat{\theta}X_A^*(\hat{\theta}_2) - I + b}{X_A^*(\hat{\theta}_2)^\beta} = \frac{\hat{\theta}X_A^*(\theta_2) - I + b}{X_A^*(\theta_2)^\beta}. \tag{50}$$

(such $\hat{\theta}$ always exists and lies between $\hat{\theta}_2$ and $\hat{\theta}_1$ for the same reason as in contract C_1). Intuitively, contract C_2 is the same as contract C, except that it substitutes a subset $\left[\hat{\theta}_2, \theta_2\right]$ of the decreasing region with a piecewise flat region with a discontinuity at $\hat{\theta}$. Because contract C is incentive-compatible and $\hat{\theta}$ satisfies (50), contract C_2 is incentive-compatible too. Below we show that the payoff to the principal from contract C_2 exceeds the payoff to the principal from contract C for $\hat{\theta}_2$ very close to θ_2. Because $\hat{X}_2(\theta) = \hat{X}(\theta)$ for $\theta \leq \theta_1$ and $\theta \geq \theta_2$, it is enough to restrict attention to the payoff in the range $\theta \in (\theta_1, \theta_2)$. The payoff to the principal from contract C_2 in this range, divided by $X(0)^\beta \frac{1}{1-\underline{\theta}}$, is

$$\int_{\theta_1}^{\hat{\theta}_2} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta} d\theta + \int_{\hat{\theta}_2}^{\hat{\theta}(\hat{\theta}_2)} \frac{\theta X_A^*(\hat{\theta}_2) - I}{X_A^*(\hat{\theta}_2)^\beta} d\theta + \int_{\hat{\theta}(\hat{\theta}_2)}^{\theta_2} \frac{\theta X_A^*(\theta_2) - I}{X_A^*(\theta_2)^\beta} d\theta. \tag{51}$$

Following the same arguments as for the derivative of (44) with respect to $\tilde{\theta}_2$ above, we can check that the derivative of (51) with respect to $\hat{\theta}_2$ is given by $\beta\frac{(\hat{\theta}-\hat{\theta}_2)^2}{2\hat{\theta}_2^2}X_A^*(\hat{\theta}_2)^{-\beta}(I+b)$, which is strictly negative at any point $\hat{\theta}_2 < \theta_2$ for $b < -I$. By the mean value theorem, if $U_P(\hat{\theta}_2)$ stands for the expected principal's utility from contract C, then $\frac{U_P(\hat{\theta}_2) - U_P(\theta_2)}{\hat{\theta}_2 - \theta_2} = U_P'\left(\tilde{\theta}_2\right) < 0$ for some $\tilde{\theta}_2 \in \left(\hat{\theta}_2, \theta_2\right)$, and hence a deviation from contract C to contract C_2 is beneficial for the principal. Hence, contract C cannot be optimal for $b < -I$. This result implies that any optimal contract must consist only of flat regions.

Step 2: *Any contract with two flat regions is dominated by a contract with one flat region.*

Consider a contract with two flat regions: Types $\left[\underline{\theta}, \hat{\theta}\right]$ pick exercise at $\hat{X}_L$, and types $\left[\hat{\theta}, 1\right]$ pick exercise at $\hat{X}_H < \hat{X}_L$. Type $\hat{\theta} \in (\underline{\theta}, 1)$ satisfies

$$\frac{\hat{\theta}\hat{X}_L - I + b}{\hat{X}_L^\beta} = \frac{\hat{\theta}\hat{X}_H - I + b}{\hat{X}_H^\beta}. \tag{52}$$

Consider an alternative contract with $\hat{X}(\theta) = \hat{X}_H$ for all θ. The difference between the principal's value under this pooling contract and her value under the original contract, divided by $X(0)^\beta$, is given by

$$\begin{aligned}\Delta U &= \int_{\underline{\theta}}^{1} \frac{\theta \hat{X}_H - I}{\hat{X}_H^\beta} \frac{d\theta}{1-\underline{\theta}} - \left[\int_{\underline{\theta}}^{\hat{\theta}} \frac{\theta \hat{X}_L - I}{\hat{X}_L^\beta} \frac{d\theta}{1-\underline{\theta}} + \int_{\hat{\theta}}^{1} \frac{\theta \hat{X}_H - I}{\hat{X}_H^\beta} \frac{d\theta}{1-\underline{\theta}}\right] = \int_{\underline{\theta}}^{\hat{\theta}} \left(\frac{\theta \hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\theta \hat{X}_L - I}{\hat{X}_L^\beta}\right) \frac{d\theta}{1-\underline{\theta}} \\ &= \frac{\hat{\theta}-\underline{\theta}}{1-\underline{\theta}} \left(\frac{\frac{\hat{\theta}+\underline{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\frac{\hat{\theta}+\underline{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^\beta}\right) = \frac{\hat{\theta}-\underline{\theta}}{1-\underline{\theta}} \left(\frac{\frac{\hat{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\frac{\hat{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^\beta} + \frac{\underline{\theta}}{2}\left(\frac{1}{\hat{X}_H^{\beta-1}} - \frac{1}{\hat{X}_L^{\beta-1}}\right)\right)\end{aligned} \tag{53}$$

Using (52) and the fact that $b \leq -I$,

$$\frac{\hat{\theta}\hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\hat{\theta}\hat{X}_L - I}{\hat{X}_L^\beta} = b\left(\frac{1}{\hat{X}_L^\beta} - \frac{1}{\hat{X}_H^\beta}\right) \geq I\left(\frac{1}{\hat{X}_H^\beta} - \frac{1}{\hat{X}_L^\beta}\right) \Leftrightarrow \frac{\frac{\hat{\theta}}{2}\hat{X}_H - I}{\hat{X}_H^\beta} - \frac{\frac{\hat{\theta}}{2}\hat{X}_L - I}{\hat{X}_L^\beta} \geq 0,$$

and the inequalities are strict if $b < -I$. Combining this with $\hat{X}_H < \hat{X}_L$ and using (53), implies that $\Delta U \geq 0$ and $\Delta U > 0$ if at least one of $b < -I$ or $\underline{\theta} > 0$ holds. Thus, the contract with two flat regions is dominated by a contract with one flat region.

Step 3: *Any contract with at least three flat regions cannot be optimal.*

The proof of this step is similar to the proof of Step 3 in Proposition 4 in Melumad and Shibano (1991) for the payoff specification in our model. Suppose, on the contrary, that the optimal contract $X(\theta)$ has at least three flat regions. Consider three adjacent steps, $X_L > X_M > X_H$, of the assumed optimal contract. Types $\left(\hat{\theta}_0, \hat{\theta}_1\right)$ pick exercise at X_L, types $\left(\hat{\theta}_1, \hat{\theta}_2\right)$ pick exercise at X_M, and types $\left(\hat{\theta}_2, \hat{\theta}_3\right)$ pick exercise at X_H, where $\underline{\theta} \leq \hat{\theta}_0 < \hat{\theta}_1 < \hat{\theta}_2 < \hat{\theta}_3 \leq 1$. Incentive compatibility implies that types $\hat{\theta}_1$ and $\hat{\theta}_2$ satisfy

$$\frac{\hat{\theta}_1 X_L - I + b}{X_L^\beta} = \frac{\hat{\theta}_1 X_M - I + b}{X_M^\beta} \Leftrightarrow \hat{\theta}_1 = \frac{(I-b)\left(X_M^{-\beta} - X_L^{-\beta}\right)}{X_M^{1-\beta} - X_L^{1-\beta}}, \tag{54}$$

$$\frac{\hat{\theta}_2 X_M - I + b}{X_M^\beta} = \frac{\hat{\theta}_2 X_H - I + b}{X_H^\beta} \Leftrightarrow \hat{\theta}_2 = \frac{(I-b)\left(X_H^{-\beta} - X_M^{-\beta}\right)}{X_H^{1-\beta} - X_M^{1-\beta}}. \tag{55}$$

Consider an alternative contract with $\tilde{X}(\theta) = X_L$ for types $(\hat{\theta}_0, y)$, $\tilde{X}(\theta) = X_H$ for types $(y, \hat{\theta}_3)$, and $\tilde{X}(\theta) = X(\theta)$ otherwise, where $y \in (\hat{\theta}_1, \hat{\theta}_2)$ satisfies

$$\frac{yX_L - I + b}{X_L^\beta} = \frac{yX_H - I + b}{X_H^\beta} \Leftrightarrow y = \frac{(I-b)\left(X_H^{-\beta} - X_L^{-\beta}\right)}{X_H^{1-\beta} - X_L^{1-\beta}}. \tag{56}$$

This contract is incentive-compatible. The difference between the principal's value under this contract and the original contract, divided by $X(0)^\beta \frac{1}{1-\underline{\theta}}$, is given by

$$\begin{aligned}&\int_{\hat{\theta}_1}^{y} \left(\frac{\theta X_L - I}{X_L^\beta} - \frac{\theta X_M - I}{X_M^\beta}\right) d\theta + \int_{y}^{\hat{\theta}_2} \left(\frac{\theta X_H - I}{X_H^\beta} - \frac{\theta X_M - I}{X_M^\beta}\right) d\theta \\ &= \left(y - \hat{\theta}_1\right)\left(\frac{\frac{y+\hat{\theta}_1}{2}X_L - I}{X_L^\beta} - \frac{\frac{y+\hat{\theta}_1}{2}X_M - I}{X_M^\beta}\right) + \left(\hat{\theta}_2 - y\right)\left(\frac{\frac{y+\hat{\theta}_2}{2}X_H - I}{X_H^\beta} - \frac{\frac{y+\hat{\theta}_2}{2}X_M - I}{X_M^\beta}\right).\end{aligned}$$

Using the left equalities of (54) and (55), we can rewrite this as

$$\left(y-\hat{\theta}_1\right)\left(\frac{y}{2}\left(\frac{X_L}{X_L^\beta}-\frac{X_M}{X_M^\beta}\right)+\frac{-I+b}{2X_M^\beta}-\frac{-I+b}{2X_L^\beta}-I\left(\frac{1}{X_L^\beta}-\frac{1}{X_M^\beta}\right)\right)$$
$$+\left(\hat{\theta}_2-y\right)\left(\frac{y}{2}\left(\frac{X_H}{X_H^\beta}-\frac{X_M}{X_M^\beta}\right)-I\left(\frac{1}{X_H^\beta}-\frac{1}{X_M^\beta}\right)+\frac{-I+b}{2X_M^\beta}-\frac{-I+b}{2X_H^\beta}\right).$$

Plugging in the values for y, $\hat{\theta}_1$, and $\hat{\theta}_2$ from the right equalities of (54), (55), and (56), we get

$$\frac{I-b}{X_M^{1-\beta}-X_L^{1-\beta}}\frac{\Sigma}{X_H^{1-\beta}-X_L^{1-\beta}}\left(\frac{\frac{(I-b)}{2}\left(X_H^{-\beta}-X_L^{-\beta}\right)\left(X_L^{1-\beta}-X_M^{1-\beta}\right)}{X_H^{1-\beta}-X_L^{1-\beta}}+\frac{b+I}{2}\left(X_M^{-\beta}-X_L^{-\beta}\right)\right)$$
$$+\frac{I-b}{X_H^{1-\beta}-X_M^{1-\beta}}\frac{\Sigma}{X_H^{1-\beta}-X_L^{1-\beta}}\left(\frac{\frac{(I-b)}{2}\left(X_H^{-\beta}-X_L^{-\beta}\right)\left(X_H^{1-\beta}-X_M^{1-\beta}\right)}{X_H^{1-\beta}-X_L^{1-\beta}}+\frac{b+I}{2}\left(X_M^{-\beta}-X_H^{-\beta}\right)\right),$$

where $\Sigma=X_H^{-\beta}X_M^{-\beta}\left(X_M-X_H\right)+X_L^{-\beta}X_H^{-\beta}\left(X_H-X_L\right)+X_M^{-\beta}X_L^{-\beta}\left(X_L-X_M\right)$. Rearranging, we obtain

$$\frac{\Sigma\left(I-b\right)\left(b+I\right)}{2(X_H^{1-\beta}-X_L^{1-\beta})}\left(\frac{X_M^{-\beta}-X_L^{-\beta}}{X_M^{1-\beta}-X_L^{1-\beta}}+\frac{X_M^{-\beta}-X_H^{-\beta}}{X_H^{1-\beta}-X_M^{1-\beta}}\right)=\frac{\frac{1}{2}\Sigma\left(I^2-b^2\right)}{(X_H^{1-\beta}-X_L^{1-\beta})}\frac{-\Sigma}{(X_M^{1-\beta}-X_L^{1-\beta})(X_H^{1-\beta}-X_M^{1-\beta})},$$

which is strictly positive because $I^2-b^2<0$ and $X_L^{1-\beta}<X_M^{1-\beta}<X_H^{1-\beta}$. Thus, contract $X\left(\theta\right)$ is strictly dominated by contract $\tilde{X}\left(\theta\right)$ and hence cannot be optimal. ■

Supplementary analysis for the proof of Proposition 1. Proof that the principal's ex-ante IC constraint is satisfied. Let $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ denote the expected value to the principal in the equilibrium with continuous exercise (up to a cutoff) if the current value of $X\left(t\right)$ is X and the current belief is that $\theta\in\left[\underline{\theta},\hat{\theta}\right]$ for some $\hat{\theta}>\hat{\theta}^*$. If the agent's type is $\theta>\hat{\theta}^*$, exercise occurs at threshold $\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, and the principal's payoff upon exercise is $\frac{\beta}{\beta-1}\left(I-b\right)-I$. If $\theta<\hat{\theta}^*$, exercise occurs at threshold X^*. Hence,

$$\left(\hat{\theta}-\underline{\theta}\right)V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)=\left(\frac{X}{X^*}\right)^\beta\int_{\underline{\theta}}^{\hat{\theta}^*}\left(\theta X^*-I\right)d\theta+X^\beta\int_{\hat{\theta}^*}^{\hat{\theta}}\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{-\beta}\left(\frac{\beta}{\beta-1}\left(I-b\right)-I\right)d\theta. \tag{57}$$

Given belief $\theta\in\left[\underline{\theta},\hat{\theta}\right]$, the principal can either wait and get $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ or exercise immediately and get $X\frac{\underline{\theta}+\hat{\theta}}{2}-I$. The current value of $X\left(t\right)$ satisfies $X\left(t\right)\leq X_A^*(\hat{\theta})$ because otherwise, the principal's belief would not be that $\theta\in\left[\underline{\theta},\hat{\theta}\right]$. Hence, the ex-ante IC condition requires that for any $\hat{\theta}>\hat{\theta}^*$, $V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\geq X\frac{\underline{\theta}+\hat{\theta}}{2}-I$ for any $X\leq X_A^*(\hat{\theta})$. Because $X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ does not depend on X, this condition is equivalent to

$$X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\geq\max_{X\in(0,X_A^*(\hat{\theta})]}\frac{1}{X^\beta}\left(X\frac{\underline{\theta}+\hat{\theta}}{2}-I\right). \tag{58}$$

The function $\frac{1}{X^\beta}\left(X\frac{\underline{\theta}+\hat{\theta}}{2}-I\right)$ is inverse U-shaped in X and has an unconditional maximum at $\frac{\beta}{\beta-1}\frac{2I}{\underline{\theta}+\hat{\theta}}$, which is strictly greater than $X_A^*(\hat{\theta})$ for any $\hat{\theta}>\frac{I-b}{I+b}\underline{\theta}=\hat{\theta}^*$. Because $X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)$ does not depend on X, (58) is equivalent to

$$X^{-\beta}V_P^c\left(X,\hat{\theta};\hat{\theta}^*\right)\geq X_A^*(\hat{\theta})^{-\beta}\left(X_A^*(\hat{\theta})\frac{\underline{\theta}+\hat{\theta}}{2}-I\right).$$

Suppose there exists $\hat{\theta}$ for which the ex-ante IC constraint is violated, i.e.,

$$\int_{\underline{\theta}}^{\hat{\theta}^*} (X^*)^{-\beta} (\theta X^* - I) \, d\theta + \int_{\hat{\theta}^*}^{\hat{\theta}} \left(\frac{\beta}{\beta-1} \frac{I-b}{\theta} \right)^{-\beta} \left(\frac{\beta}{\beta-1} (I-b) - I \right) d\theta < \left(\hat{\theta} - \underline{\theta} \right) X_A^*(\hat{\theta})^{-\beta} \left(X_A^*(\hat{\theta}) \frac{\underline{\theta} + \hat{\theta}}{2} - I \right). \tag{59}$$

We show that this implies that the contract derived in Lemma 1 cannot be optimal, which is a contradiction. In particular, denote the contract from the second part of Lemma 1 by $\hat{X}_-(\theta)$. Then (59) implies that the contract $\hat{X}_-(\theta)$ is dominated by the contract with continuous exercise at $X_A^*(\theta)$ for $\theta \geq \hat{\theta}$ and exercise at $X_A^*(\hat{\theta})$ for $\theta \leq \hat{\theta}$. Indeed, the principal's expected utility under the contract $\hat{X}_-(\theta)$, divided by $X(0)^{\beta}$, is

$$\int_{\underline{\theta}}^{\hat{\theta}^*} (X^*)^{-\beta} (\theta X^* - I) \, d\theta + \int_{\hat{\theta}^*}^{1} \left(\frac{\beta}{\beta-1} \frac{I-b}{\theta} \right)^{-\beta} \left(\frac{\beta}{\beta-1} (I-b) - I \right) d\theta. \tag{60}$$

Similarly, the principal's expected utility under the modified contract (also divided by $X(0)^{\beta}$), where $\frac{I-b}{I+b}\underline{\theta}$ in $\hat{X}_-(\theta)$ is replaced by $\hat{\theta}$, and the cutoff $\frac{\beta}{\beta-1}\frac{I+b}{\underline{\theta}}$ in $\hat{X}_-(\theta)$ is replaced by $X_A^*(\hat{\theta}) = \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}}$, is given by

$$\int_{\underline{\theta}}^{\hat{\theta}} X_A^*(\hat{\theta})^{-\beta} \left(\theta X_A^*(\hat{\theta}) - I \right) d\theta + \int_{\hat{\theta}}^{1} \left(\frac{\beta}{\beta-1} \frac{I-b}{\theta} \right)^{-\beta} \left(\frac{\beta}{\beta-1} (I-b) - I \right) d\theta. \tag{61}$$

Combining (60) and (61), it is easy to see that the contract with continuous exercise up to the cutoff $\hat{\theta}$ dominates the contract $\hat{X}_-(\theta)$ if and only if (59) is satisfied. Hence, the ex-ante IC constraint is indeed satisfied. ■

Supplementary analysis for the proof of Proposition 2. Part 1. Derivation of the principal's value function in the ω-equilibrium, $V_P(X(t), 1; \omega)$.

The principal's value function $V_P(X(t), 1; \omega)$ satisfies

$$rV_P(X, 1; \omega) = \alpha X V_{P,X}(X, 1; \omega) + \frac{1}{2}\sigma^2 X^2 V_{P,XX}(X, 1; \omega). \tag{62}$$

The value matching condition is:

$$V_P(Y(\omega), 1; \omega) = \int_{\omega}^{1} (\theta Y(\omega) - I) \, d\theta + \omega V_P(Y(\omega), \omega; \omega). \tag{63}$$

The intuition behind (63) is as follows. With probability $1-\omega$, θ is above ω. In this case, the agent recommends exercise, and the principal follows the recommendation. The payoff of the principal, given θ, is $\theta Y(\omega) - I$. With probability ω, θ is below ω, so the agent recommends against exercise, and the option is not exercised. The continuation payoff of the principal in this case is $V_P(Y(\omega), \omega; \omega)$. Solving (62) subject to (63), we obtain

$$V_P(X, 1; \omega) = \left(\frac{X}{Y(\omega)} \right)^{\beta} \left(\int_{\omega}^{1} (\theta Y(\omega) - I) \, d\theta + \omega V_P(Y(\omega), \omega; \omega) \right). \tag{64}$$

By stationarity,

$$V_P(Y(\omega), \omega; \omega) = V_P(\omega Y(\omega), 1; \omega). \tag{65}$$

Evaluating (64) at $X = \omega Y(\omega)$ and using the stationarity condition (65), we obtain:

$$V_P(\omega Y(\omega), 1; \omega) = \omega^\beta \left[\frac{1}{2} \left(1-\omega^2\right) Y(\omega) - (1-\omega) I \right] + \omega^{\beta+1} V_P(\omega Y(\omega), 1; \omega).$$

Therefore,

$$V_P(\omega Y(\omega), 1; \omega) = \frac{\omega^\beta (1-\omega)}{1-\omega^{\beta+1}} \left[\frac{1}{2} (1+\omega) Y(\omega) - I \right]. \quad (66)$$

Plugging (66) into (64), we obtain the principal's value function (16).

Part 2. Existence of ω-equilibria for $b < 0$.

2a. Proof that $G(\omega) > 0$, where $G(\omega) \equiv \frac{\left(1-\omega^\beta\right)(I-b)}{\omega\left(1-\omega^{\beta-1}\right)} - \frac{\beta}{\beta-1}\frac{2(I-b)}{1+\omega}$. Note that $G(\omega) = \frac{2(I-b)}{1+\omega} g(\omega)$, where $g(\omega) \equiv \frac{\left(1-\omega^\beta\right)(1+\omega)}{2\left(\omega-\omega^\beta\right)} - \frac{\beta}{\beta-1}$. We have:

$$\lim_{\omega\to 1} g(\omega) = \lim_{\omega\to 1} \frac{1-\omega^\beta - \beta\omega^{\beta-1}(1+\omega)}{2\left(1-\beta\omega^{\beta-1}\right)} - \frac{\beta}{\beta-1} = 0,$$
$$g'(\omega) = \frac{\beta\left(\omega^{\beta-1}-\omega^{\beta+1}\right)+\omega^{2\beta}-1}{2\left(\omega-\omega^\beta\right)^2},$$

where the first limit holds by l'Hopital's rule. Denote the numerator of $g'(\omega)$ by $h(\omega) \equiv \omega^{2\beta} - \beta\omega^{\beta+1} + \beta\omega^{\beta-1} - 1$. Function $h(\omega)$ is a generalized polynomial. By an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883),[21] the number of positive roots of $h(\omega)$, counted with their orders, does not exceed the number of sign changes of coefficients of $h(\omega)$, i.e., three. Because $\omega = 1$ is the root of $h(\omega)$ of order three and $h(0) < 0$, then $h(\omega) < 0$ for all $\omega \in [0,1)$, and hence $g'(\omega) < 0$ for all $\omega \in [0,1)$. Combined with $\lim_{\omega\to 1} g(\omega) = 0$, this implies $g(\omega) > 0$ and hence $G(\omega) > 0$ for all $\omega \in [0,1)$.

2b. Proof of Step 1: *If $b < 0$, $V_P(X, 1; \omega)$ is strictly increasing in ω for any $\omega \in (0,1)$.*

Because $V_P(X, 1; \omega)$ is proportional to X^β, it is enough to prove the statement for $X = 1$. We can re-write $V_P(1, 1; \omega)$ as $2^{-\beta} f_1(\omega) f_2(\omega)$, where

$$f_1(\omega) \equiv \frac{(1-\omega)(1+\omega)^\beta}{1-\omega^{\beta+1}} \text{ and } f_2(\omega) \equiv \frac{\frac{1}{2}(1+\omega) Y(\omega) - I}{\left(\frac{1}{2}(1+\omega) Y(\omega)\right)^\beta}. \quad (67)$$

Since, as shown above, $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for $\omega < 1$, then $\frac{1}{2}(1+\omega) Y(\omega) > \frac{\beta}{\beta-1} I > I$, and hence $f_2(\omega) > 0$ for $\omega < 1$. Because $f_1(\omega) > 0$ and $f_2(\omega) > 0$ for any $\omega \in (0, \omega^*)$, a sufficient condition for $V_P(1, 1; \omega)$ to be increasing is that both $f_1(\omega)$ and $f_2(\omega)$ are increasing for $\omega \in (0, \omega^*)$.

First, consider $f_2(\omega)$. As an auxiliary result, we prove that $(1+\omega) Y(\omega)$ is strictly decreasing in ω. This follows from the fact that

$$\frac{\partial\left((1+\omega) Y(\omega)\right)}{\partial\omega} = (I-b)\frac{-1+\beta\omega^{\beta-1}-\beta\omega^{\beta+1}+\omega^{2\beta}}{\left(\omega-\omega^\beta\right)^2}$$

and that as shown above, the numerator, $h(\omega)$, is strictly negative for all $\omega \in [0,1)$. Next,

$$f_2'(\omega) = \frac{(\beta-1)(1+\omega)}{4\left(\frac{1}{2}(1+\omega) Y(\omega)\right)^{\beta+1}} \left(\frac{\beta}{\beta-1}\frac{2I}{1+\omega} - Y(\omega) \right) \frac{\partial\left((1+\omega) Y(\omega)\right)}{\partial\omega}.$$

[21] See Theorem 3.1 in Jameson (2006).

Because $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for $\omega < 1$ as shown above, and because $(1+\omega)Y(\omega)$ is strictly decreasing in ω, $f_2'(\omega) > 0$ for any $\omega \in (0, \omega^*)$.

Second, consider $f_1(\omega)$. Note that

$$f_1'(\omega) = \frac{(1+\omega)^{\beta-1}}{1-\omega^{\beta+1}} \frac{\beta - 1 - (\beta+1)\omega + (\beta+1)\omega^{\beta} - (\beta-1)\omega^{\beta+1}}{1-\omega^{\beta+1}}.$$

Denote the numerator of the second fraction by $d(\omega) \equiv -(\beta-1)\omega^{\beta+1} + (\beta+1)\omega^{\beta} - (\beta+1)\omega + \beta - 1$. By an extension of Descartes' Rule of Signs to generalized polynomials, the number of positive roots of $d(\omega)$ does not exceed the number of sign changes of coefficients of $d(\omega)$, i.e., three. It is easy to show that $d(1) = d'(1) = d''(1) = 0$. Hence, $\omega = 1$ is the root of $d(\omega) = 0$ of order three, and $d(\omega)$ does not have roots other than $\omega = 1$. Since $d(0) = \beta - 1 > 0$, this implies that for any $\omega \in (0,1)$, $d(\omega) > 0$. Hence, $f_1'(\omega) > 0$, which completes the proof of this step.

2c. Proof of Step 2: $\lim_{\omega\to 1} V_P(X,1;\omega) = V_P^c(X,1)$.

According to (67), $V_P(X,1;\omega) = 2^{-\beta}X^{\beta} f_1(\omega) f_2(\omega)$. By l'Hopital's rule, $\lim_{\omega\to 1} f_1(\omega) = \frac{2^{\beta}}{\beta+1}$, $\lim_{\omega\to 1} Y(\omega) = \frac{\beta}{\beta-1}(I-b)$, and hence $\lim_{\omega\to 1} f_1(\omega) = (\frac{\beta}{\beta-1}(I-b) - I)(\frac{\beta}{\beta-1}(I-b))^{-\beta}$.Using (10), it is easy to see that $\lim_{\omega\to 1} V_P(X,1;\omega) = 2^{-\beta}X^{\beta}\lim_{\omega\to 1} f_1(\omega)\lim_{\omega\to 1} f_1(\omega) = V_P^c(X,1)$.

2d. Proof of Step 3. *Suppose $-I < b < I$. For ω close enough to zero, the ex-ante IC condition (20) does not hold.*

The function $X^{-\beta}\left(\frac{1}{2}X - I\right)$ is inverse U-shaped and has a maximum at $\bar{X}_u = \frac{\beta}{\beta-1}2I$. When ω is close to zero, $Y(\omega) = \frac{(1-\omega^{\beta})(I-b)}{\omega(1-\omega^{\beta-1})} \to +\infty$, and hence $\max_{X\in(0,Y(\omega)]} X^{-\beta}\left(\frac{1}{2}X - I\right) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$. Hence, we can rewrite (20) as $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$, and it is easy to show that it is equivalent to

$$\left(\omega - \omega^{\beta}\right)^{\beta-1} H(\omega) \geq (I-b)^{\beta}\left(1-\omega^{\beta+1}\right)\left(1-\omega^{\beta}\right)^{\beta}, \tag{68}$$

where

$$H(\omega) \equiv 2^{\beta-1}\beta^{\beta}\left(\frac{I}{\beta-1}\right)^{\beta-1}(1-\omega)\left(I(1-\omega)\left(1+\omega^{\beta}\right) - b(1+\omega)\left(1-\omega^{\beta}\right)\right).$$

Since $H(0) > 0$, then as $\omega \to 0$, the left-hand side of (68) converges to zero, while the right-hand side converges to $(I-b)^{\beta} > 0$. Hence, for ω close enough to 0, the ex-ante IC condition is violated.

2e. Proof of Step 4. *Suppose $-I < b < I$. Then (20) is satisfied for any $\omega \geq \bar{\omega}$, where $\bar{\omega}$ is the unique solution to $Y(\omega) = \bar{X}_u$. For any $\omega < \bar{\omega}$, (20) is satisfied if and only if $X^{-\beta}V_P(X,1;\omega) \geq \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$.*

Note that for any $b > -I$, $\lim_{\omega\to 1} Y(\omega) = \frac{\beta(I-b)}{\beta-1} < \frac{\beta}{\beta-1}2I = \bar{X}_u$, and hence there exists a unique $\bar{\omega}$ such that $Y(\omega) \leq \bar{X}_u \Leftrightarrow \omega \geq \bar{\omega}$. Hence, (20) becomes

$$X^{-\beta}V_P(X,1;\omega) \quad \geq \quad \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right) \text{ for } \omega \leq \bar{\omega}, \tag{69}$$

$$X^{-\beta}V_P(X,1;\omega) \quad \geq \quad Y(\omega)^{-\beta}\left(\frac{1}{2}Y(\omega) - I\right) \text{ for } \omega \geq \bar{\omega}. \tag{70}$$

Suppose that (70) is satisfied for some $\tilde{\omega} \geq \bar{\omega}$. Because $Y(\omega)$ is decreasing, $Y(\tilde{\omega}) \geq Y(\omega)$ for $\omega \geq \tilde{\omega}$. Because $X^{-\beta}\left(\frac{1}{2}X - I\right)$ is increasing for $X \leq \bar{X}_u$ and because $Y(\omega) \leq Y(\tilde{\omega}) \leq Y(\bar{\omega}) = \bar{X}_u$ for $\omega \geq \tilde{\omega} \geq \bar{\omega}$, we have $Y(\omega)^{-\beta}\left(\frac{1}{2}Y(\omega) - I\right) \leq Y(\tilde{\omega})^{-\beta}\left(\frac{1}{2}Y(\tilde{\omega}) - I\right)$ for any $\omega \geq \tilde{\omega}$. On the other hand, according to Step 1, $X^{-\beta}V_P(X,1;\tilde{\omega}) \leq X^{-\beta}V_P(X,1;\omega)$ for any $\omega \geq \tilde{\omega}$. Hence, if (70) is satisfied for $\tilde{\omega} \geq \bar{\omega}$, it is also satisfied for any $\omega \in [\tilde{\omega}, 1)$. Hence, to prove that (20) is satisfied for any $\omega \geq \bar{\omega}$, it is sufficient to prove (70) for $\omega = \bar{\omega}$. Using (16) and the fact that $Y(\bar{\omega}) = \bar{X}_u$, (70) for $\omega = \bar{\omega}$ is equivalent to

$$\begin{aligned}
\frac{1-\bar{\omega}}{1-\bar{\omega}^{\beta+1}} \bar{X}_u^{-\beta} \left(\frac{1}{2} (1+\bar{\omega}) \bar{X}_u - I \right) &\geq \bar{X}_u^{-\beta} \left(\frac{1}{2} \bar{X}_u - I \right) \Leftrightarrow \frac{1}{2} \bar{X}_u \left(\frac{1-\bar{\omega}^2}{1-\bar{\omega}^{\beta+1}} - 1 \right) \geq I \left(\frac{1-\bar{\omega}}{1-\bar{\omega}^{\beta+1}} - 1 \right) \\
&\Leftrightarrow \frac{1}{2I} \bar{X}_u \leq \frac{\bar{\omega} - \bar{\omega}^{\beta+1}}{\bar{\omega}^2 - \bar{\omega}^{\beta+1}} \Leftrightarrow \frac{\beta}{\beta-1} \leq \frac{\bar{\omega} - \bar{\omega}^{\beta+1}}{\bar{\omega}^2 - \bar{\omega}^{\beta+1}} \qquad (71)
\end{aligned}$$

Consider the function $Q(\omega) \equiv \frac{\omega - \omega^{\beta+1}}{\omega^2 - \omega^{\beta+1}}$. Note that $Q'(\omega) < 0 \Leftrightarrow q(\omega) \equiv (\beta-1)\omega^\beta - \beta\omega^{\beta-1} + 1 > 0$. By an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883), the number of positive roots of $q(\omega)$, counted with their orders, does not exceed the number of sign changes of coefficients of $q(\omega)$, i.e., two. Since $q(1) = q'(1) = 0$, $q(\omega)$ does not have any roots on $(0, \infty)$ other than 1. Since $q''(1) > 0$, we have $q(\omega) > 0$ for all $\omega \in (0,1)$, and hence $Q'(\omega) < 0$. By l'Hopital's rule, $\lim_{\omega \to 1} Q(\omega) = \frac{\beta}{\beta-1}$, and hence $\frac{\beta}{\beta-1} \leq Q(\omega)$ for any $\omega \in (0,1)$, which proves (71).

Part 3. Existence of ω-equilibria for $b > 0$.

3a. Proof that in the range $[0,1]$, equation $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$ has a unique solution $\omega^* \in (0,1)$, where ω^* decreases in b, $\lim_{b\to 0}\omega^* = 1$, and $\lim_{b \to I}\omega^* = 0$.

The equality $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{\omega+1}$ is equivalent to

$$\omega = \frac{1}{\frac{\beta}{\beta-1}\frac{1-\omega^{\beta-1}}{1-\omega^\beta}\frac{2I}{I-b} - 1}. \qquad (72)$$

We next show that if $0 < b < I$, then in the range $[0,1]$, equation (72) has a unique solution $\omega^* \in (0,1)$, where ω^* decreases in b, $\lim_{b\to 0}\omega^* = 1$, and $\lim_{b\to I}\omega^* = 0$. We can rewrite (72) as

$$\omega = \frac{(\beta-1)\left(1-\omega^\beta\right)(I-b)}{\beta\left(1-\omega^{\beta-1}\right)2I - (\beta-1)\left(1-\omega^\beta\right)(I-b)} \Leftrightarrow \frac{2\beta I\left(\omega - \omega^\beta\right) + (\beta-1)(I-b)\left(\omega^{\beta+1} - \omega - 1 + \omega^\beta\right)}{\beta\left(1-\omega^{\beta-1}\right)2I - (\beta-1)\left(1-\omega^\beta\right)(I-b)} = 0.$$

Denote the left-hand side of the second equation as a function of ω by $l(\omega)$. The denominator of $l(\omega)$, $l_d(\omega)$, is nonnegative on $\omega \in [0,1]$ and equals zero only at $\omega = 1$. This follows from $l_d(0) = 2\beta I - (\beta-1)(I-b) > 0$, $l_d(1) = 0$, and $l_d'(\omega) = \beta(\beta-1)\omega^{\beta-2}(-2I + \omega(I-b)) < 0$.Therefore, $l(\omega) = 0$ if and only if the numerator of $l(\omega)$, $l_n(\omega)$, equals zero at $\omega \in (0,1)$. Since $b \in (0, I)$, then $l_n(0) = -(\beta-1)(I-b) < 0$,

$$\begin{aligned}
l_n'(\omega) &= 2\beta I\left(1-\beta\omega^{\beta-1}\right) + (\beta-1)(I-b)\left((\beta+1)\omega^\beta - 1 + \beta\omega^{\beta-1}\right), \\
l_n''(\omega) &= -2\beta^2(\beta-1)I\omega^{\beta-2} + (\beta-1)(I-b)\left(\beta(\beta+1)\omega^{\beta-1} + \beta(\beta-1)\omega^{\beta-2}\right),
\end{aligned}$$

and

$$l_n''(\omega) < 0 \Leftrightarrow (I-b)((\beta+1)\omega + \beta - 1) < 2\beta I \Leftrightarrow \omega < \frac{(\beta+1)I + (\beta-1)b}{(\beta+1)(I-b)}.$$

Since $\frac{(\beta+1)I+(\beta-1)b}{(\beta+1)(I-b)} > 1$, $l_n''(\omega) < 0$ for any $\omega \in [0,1]$. Since $l_n'(0) = 2\beta I - (\beta-1)(I-b) > 0$ and $l_n'(1) = -2\beta(\beta-1)b < 0$, there exists $\hat{\omega} \in (0,1)$ such that $l_n(\omega)$ increases to the left of $\hat{\omega}$ and decreases to the right. Since $\lim_{\omega\to 1} l_n(\omega) = 0$, then $l_n(\hat{\omega}) > 0$, and hence $l_n(\omega)$ has a unique root ω^* on $(0,1)$.

Since the function $l_n(\omega)$ increases in b and is strictly increasing at the point ω^*, then ω^* decreases in b. To prove that $\lim_{b\to 0}\omega^* = 1$, it is sufficient to prove that for any small $\varepsilon > 0$, there exists $b(\varepsilon) > 0$ such that $l_n(1-\varepsilon) < 0$ for $b < b(\varepsilon)$. Since $l_n(\omega) > 0$ on $(\omega^*, 1)$, this would imply that $\omega^* \in (1-\varepsilon, 1)$, i.e., that

ω^* is infinitely close to 1 when b is close to zero. Using the expression for $l_n(\omega)$, $l_n(\omega) < 0$ is equivalent to

$$\frac{2\beta}{\beta-1}\frac{\omega}{\omega+1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}} < 1-\frac{b}{I}. \tag{73}$$

Denote the left-hand side of (73) by $L(\omega)$. Note that $L(\omega)$ is increasing on $(0,1)$. Indeed, differentiating $L(\omega)$ and simplifying, $L'(\omega) > 0 \Leftrightarrow \Lambda(\omega) \equiv 1-\omega^{2\beta}-\beta\omega^{\beta-1}+\beta\omega^{\beta+1} > 0$. The function $\Lambda(\omega)$ is decreasing on $(0,1)$ because $\Lambda'(\omega) < 0 \Leftrightarrow \varphi(\omega) \equiv -2\omega^{\beta+1}-(\beta-1)+(\beta+1)\omega^2 < 0$, where $\varphi'(\omega) > 0$ and $\varphi(1) = 0$. Since $\Lambda(\omega)$ is decreasing and $\Lambda(1) = 0$, then, indeed, $\Lambda(\omega) > 0$ and hence $L'(\omega) > 0$ for all $\omega \in (0,1)$. In addition, by l'Hopital's rule, $\lim_{\omega\to 1} L(\omega) = 1$. Hence, $L(1-\varepsilon) < 1$ for any $\varepsilon > 0$, and thus $l_n(1-\varepsilon) < 0$ for $b \in [0, I(1-L(1-\varepsilon)))$.

Finally, to prove that $\lim_{b\to I}\omega^* = 0$, it is sufficient to prove that for any small $\varepsilon > 0$, there exists $b(\varepsilon)$ such that $l_n(\varepsilon) > 0$ for $b > b(\varepsilon)$. Since $l_n(0) < 0$, this would imply that $\omega^* \in (0,\varepsilon)$ for $b > b(\varepsilon)$, i.e., that ω^* is infinitely close to zero when b is close to I. Based on (73), $l_n(\omega) > 0 \Leftrightarrow L(\omega) > 1-\frac{b}{I}$. Then, for any $\varepsilon > 0$, if $b > I(1-L(\varepsilon))$, we get $1-\frac{b}{I} < L(\varepsilon) \Leftrightarrow l_n(\varepsilon) > 0$, which completes the proof.

3b. Proof that $Y(\omega)$ is strictly decreasing in ω for $\omega \in (0,1)$. Note that

$$\frac{\partial Y(\omega)}{\partial \omega} = \frac{(I-b)}{\omega(\omega-\omega^{\beta})^2}\left[-(\beta-1)\omega^{\beta+1}+\beta\omega^{\beta}-\omega\right],$$

where $\frac{(I-b)}{\omega(\omega-\omega^{\beta})^2} > 0$. Thus, we need to show that $k(\omega) \equiv -(\beta-1)\omega^{\beta+1}+\beta\omega^{\beta}-\omega < 0$. According to an extension of Descartes' Rule of Signs to generalized polynomials (Laguerre, 1883), the number of positive roots of $k(\omega) = 0$, counted with their orders, does not exceed the number of change of signs of its coefficients, i.e., two. Since $k(1) = 0$, $k'(1) = 0$, and $k"(1) = -\beta(\beta-1) < 0$, $\omega = 1$ is a root of order two, and there are no other positive roots. Further, $k(0) = 0$ and $k'(0) = -1 < 0$. It follows that $k(0) = k(1) = 0$ and $k(\omega) < 0$ for all $\omega \in (0,1)$, and hence, indeed, $\frac{\partial Y(\omega)}{\partial\omega} < 0$.

3c. Proof of Step 5: *If $b > 0$, $V_P(X,1;\omega)$ is strictly increasing in ω for any $\omega \in (0,\omega^*)$.*

The proof of this step is the same as the proof of Step 1 for the case $b < 0$ with the only difference: instead of relying on the inequality $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for all $\omega \in (0,1)$ as for the case $b < 0$ (which holds for $b < 0$), we rely on the inequality $Y(\omega) > \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$ for all $\omega \in (0,\omega^*)$.

3d. Proof of Step 6: *If $0 < b < I$, then the ex-ante IC condition (20) holds as a strict inequality for $\omega = \omega^*$.*

Using (16) and $Y(\omega) = \frac{\beta}{\beta-1}\frac{2I}{1+\omega}$, we can rewrite $V_P(X,1;\omega^*)$ as $X^{\beta}K(\omega^*)$, where

$$K(\omega) \equiv \frac{1-\omega}{1-\omega^{\beta+1}}\left(\frac{\beta}{\beta-1}\frac{2I}{\omega+1}\right)^{-\beta}\frac{I}{\beta-1}.$$

Note that $K(0) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right)$ and that

$$K'(\omega) > 0 \Leftrightarrow \kappa(\omega) \equiv -(\beta-1)\omega^{\beta+1}+(\beta+1)\omega^{\beta}-(\beta+1)\omega+\beta-1 > 0.$$

By an extension of Descartes' Rule of Signs to generalized polynomials, the number of positive roots of $\kappa(\omega)$, counted with their orders, does not exceed the number of change of signs of its coefficients, i.e., three. Note that $\omega = 1$ is the root of $\kappa(\omega)$ of order three: $\kappa(1) = \kappa'(1) = \kappa''(1) = 0$, and hence there are no other roots. Since $\kappa(0) = \beta - 1 > 0$, it follows that $\kappa(\omega) > 0$ and hence $K'(\omega) > 0$ for all $\omega \in [0,1)$. Therefore,

$K(\omega)$ is strictly increasing in ω, which implies

$$X^{-\beta}V_P(X,1;\omega^*) = K(\omega^*) > K(0) = \bar{X}_u^{-\beta}\left(\frac{1}{2}\bar{X}_u - I\right). \tag{74}$$

Because the function $X^{-\beta}\left(\frac{1}{2}X - I\right)$ achieves its global maximum at the point $\bar{X}_u$, (74) implies that (20) holds as a strict inequality for $\omega = \omega^*$, which completes the proof of this step. ■

Supplementary analysis for the proof of Proposition 4. Proof that $F_\beta(\omega,\beta) < 0$. Differentiating $F(\omega,\beta)$ with respect to β and reorganizing the terms, we obtain that $F_\beta(\omega,\beta) < 0$ is equivalent to

$$\frac{\left(1-\omega^{\beta-1}\right)\left(1-\omega^{\beta}\right)}{\omega^{\beta-1}(1-\omega)} + \beta(\beta-1)\ln\omega > 0.$$

Denote the left-hand side as a function of β by $N(\beta)$. Because $N(1) = 0$, a sufficient condition for $N(\beta) > 0$ for any $\beta > 1$ is that $N'(\beta) > 0$ for $\beta > 1$. Differentiating $N(\beta)$:

$$N'(\beta) = \ln\omega\left[-\frac{\omega^{1-\beta}-\omega^{\beta}}{1-\omega} + 2\beta - 1\right].$$

Because $\ln\omega < 0$ for any $\omega \in (0,1)$, condition $N'(\beta) > 0$ is equivalent to $n(\beta) \equiv \frac{\omega^{1-\beta}-\omega^{\beta}}{1-\omega} - 2\beta + 1 > 0$. Note that $\lim_{\beta\to1} n(\beta) = 0$ and $n'(\beta) = -\left(\omega^{1-\beta}+\omega^{\beta}\right)\frac{\ln\omega}{1-\omega} - 2 \equiv \eta(\beta)$. Note that

$$\eta(\beta) = \eta(1) + \int_1^\beta \eta'(x)\,dx = -\frac{(1+\omega)\ln\omega}{1-\omega} - 2 + \frac{(\ln\omega)^2}{1-\omega}\int_1^\beta\left(\left(\frac{1}{\omega}\right)^{2x-1} - 1\right)\omega^x dx. \tag{75}$$

The second term of (75) is positive, because $\left(\frac{1}{\omega}\right)^{2x-1} - 1 > 0$, since $\frac{1}{\omega} > 1$ and $2x - 1 > 1$ for any $x > 1$. The first term of (75) is positive, because

$$\begin{aligned}&\lim_{\omega\to1}\left(-\tfrac{(1+\omega)\ln\omega}{1-\omega} - 2\right) = \lim_{\omega\to1}\left(\ln\omega + \tfrac{1+\omega}{\omega}\right) - 2 = 0\\ &\text{and } \tfrac{\partial\left(-\frac{(1+\omega)\ln\omega}{1-\omega}-2\right)}{\partial\omega} = \tfrac{-2\ln\omega - \frac{1}{\omega} + \omega}{(1-\omega)^2} < 0,\end{aligned}$$

where the first row is by l'Hopital's rule, and the second row is because $\left(-2\ln\omega - \frac{1}{\omega} + \omega\right)' = \frac{(1-\omega)^2}{\omega^2} > 0$ and $-2\ln\omega - \frac{1}{\omega} + \omega$ equals zero at $\omega = 1$. Thus, $\eta(\beta) > 0$ and hence $n'(\beta) > 0$ for any $\beta > 1$, which together with $n(1) = 0$ implies $n(\beta) > 0$, which in turn implies that $N(\beta) > 0$ for any $\beta > 1$. Hence, $F_\beta(\omega,\beta) < 0$. ■

Supplementary analysis for the proof of Proposition 6. Proof that $\lim_{b\to0}\frac{VD'(b)}{b} = -\frac{\beta^2}{\beta^2-1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta-1}$. By l'Hopital's rule,

$$\lim_{b\to0+} VD(b) = \lim_{b\to0+} VA(b) = \frac{1}{\beta+1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta}\frac{I}{\beta-1}.$$

Note that $VD'(b) = -\frac{\beta b}{(\beta+1)(I-b)}\left(\frac{\beta}{\beta-1}(I-b)\right)^{-\beta}$. In particular, $\lim_{b\to0+} VD'(b) = 0$ and $\lim_{b\to0}\frac{VD'(b)}{b} = -\frac{\beta^2}{\beta^2-1}\left(\frac{\beta}{\beta-1}I\right)^{-\beta-1}$.

Proof that $\lim_{b\to 0}\frac{VA'(b)}{b}=-\infty$. The derivative of $VA(b)$ with respect to b can be found as

$$VA'(b)=C\frac{d\omega^*(b)}{db}\left[\frac{(1-\omega)(1+\omega)^{\beta}}{1-\omega^{\beta+1}}\right]'|_{\omega=\omega^*(b)}, \tag{76}$$

where $C\equiv\left(\frac{\beta}{\beta-1}2I\right)^{-\beta}\frac{I}{\beta-1}$. Recall that $\omega^*(b)$ solves (72), which is equivalent to

$$\frac{2I}{I-b}\frac{\beta}{\beta-1}=\left(\frac{1}{\omega}+1\right)\frac{1-\omega^{\beta}}{1-\omega^{\beta-1}}. \tag{77}$$

Differentiating this equation, we get

$$\frac{2I}{(I-b)^2}\frac{\beta}{\beta-1}db=\frac{-\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)+(1+\omega)\,\omega\left(-\beta\omega^{\beta-1}\left(1-\omega^{\beta-1}\right)+(\beta-1)\,\omega^{\beta-2}\left(1-\omega^{\beta}\right)\right)}{\omega^2\left(1-\omega^{\beta-1}\right)^2}d\omega. \tag{78}$$

Because (77) is equivalent to $\frac{1}{I-b}=\frac{1}{2I}\frac{\beta-1}{\beta}\frac{1+\omega}{\omega}\frac{1-\omega^{\beta}}{1-\omega^{\beta-1}}$, we can rewrite the left-hand side of (78) as

$$2I\frac{\beta}{\beta-1}\left(\frac{1}{2I}\right)^2\left(\frac{\beta-1}{\beta}\right)^2\frac{(1+\omega)^2}{\omega^2}\frac{\left(1-\omega^{\beta}\right)^2}{\left(1-\omega^{\beta-1}\right)^2}db.$$

Substituting this into (78) and simplifying, we get

$$\frac{d\omega}{db}|_{\omega=\omega^*(b)}=\frac{1}{2I}\frac{\beta-1}{\beta}\frac{(1+\omega)^2\left(1-\omega^{\beta}\right)^2}{-\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)+(1+\omega)\,\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)}. \tag{79}$$

Plugging (79) and

$$\left[\frac{(1-\omega)(1+\omega)^{\beta}}{1-\omega^{\beta+1}}\right]'=\frac{(1+\omega)^{\beta-1}}{\left(1-\omega^{\beta+1}\right)^2}\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^{\beta}\right)\right],$$

into (76), we get

$$VA'(b)=-D\frac{(1+\omega)^{\beta+1}\left(1-\omega^{\beta}\right)^2\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^{\beta}\right)\right]}{\left(1-\omega^{\beta+1}\right)^2\left[\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)-(1+\omega)\,\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)\right]},$$

where $D\equiv\frac{C}{2I}\frac{\beta-1}{\beta}$. To find $\lim_{b\to 0}\frac{VA'(b)}{b}$, we express $\frac{1}{b}$ from (77) as

$$\frac{1}{b}=\frac{(\beta-1)(1+\omega)\left(1-\omega^{\beta}\right)}{I\left[(\beta-1)(1+\omega)\left(1-\omega^{\beta}\right)-2\beta\omega\left(1-\omega^{\beta-1}\right)\right]},$$

and hence

$$\begin{aligned}\frac{VA'(b)}{b}&=-D\frac{(1+\omega)^{\beta+1}\left(1-\omega^{\beta}\right)^2\left[(\beta-1)\left(1-\omega^{\beta+1}\right)-(\beta+1)\left(\omega-\omega^{\beta}\right)\right]}{\left(1-\omega^{\beta+1}\right)^2\left[\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)-(1+\omega)\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)\right]}\frac{(\beta-1)(1+\omega)\left(1-\omega^{\beta}\right)}{I\left[(\beta-1)(1+\omega)\left(1-\omega^{\beta}\right)-2\beta\omega\left(1-\omega^{\beta-1}\right)\right]}\\&=-\frac{(\beta-1)D}{I}\frac{(1+\omega)^{\beta+2}\left(1-\omega^{\beta}\right)^3}{\left(1-\omega^{\beta+1}\right)^2\left[\left(1-\omega^{\beta}\right)\left(1-\omega^{\beta-1}\right)-(1+\omega)\omega^{\beta-1}\left(-\beta\omega+\beta-1+\omega^{\beta}\right)\right]}.\end{aligned}$$

Hence,

$$\lim_{b\to 0}\frac{VA'(b)}{b}=-\frac{(\beta-1)\,2^{\beta+2}D}{I}\lim_{\omega\to 1}\left[\frac{1-\omega^{\beta}}{1-\omega^{\beta+1}}\right]^{2}\lim_{\omega\to 1}\left[\frac{1-\omega^{\beta}}{(1-\omega^{\beta})(1-\omega^{\beta-1})-(1+\omega)\,\omega^{\beta-1}(-\beta\omega+\beta-1+\omega^{\beta})}\right].$$

By l'Hopital's rule, the first limit equals $(\frac{\beta}{\beta+1})^2$, and the second limit equals ∞, which completes the proof. ■

Proof of Proposition 8. Consider a problem in which the agent is free to choose whether to exercise the option prior to the arrival of the news, while the principal makes the exercise decision after the arrival of the news. Let $V_A^a(X,\theta)$ be the value of the option to the agent of type θ after the arrival of the news:

$$V_A^a(X,\theta)=\begin{cases}\left(\frac{X}{X_P^*(\theta)}\right)^{\beta}(\theta X_P^*(\theta)-I+b), & \text{if } X\le X_P^*(\theta),\\ \theta X-I+b, & \text{if } X\ge X_P^*(\theta).\end{cases}$$

Let $V_A^b(X,\theta)$ be the value of the option to the agent before the arrival of the news. Since the expected return from holding an option over a small interval $[t,t+dt]$ must be rdt, $V_A^b(X,\theta)$ satisfies

$$(r+\lambda)V_A^b(X,\theta)=\alpha X\frac{\partial V_A^b(X,\theta)}{\partial X}+\frac{1}{2}\sigma^2X^2\frac{\partial^2 V_A^b(X,\theta)}{\partial X^2}+\lambda V_A^a(X,\theta). \tag{80}$$

First, we show that waiting is optimal in the range $X\le X_P^*(\theta)$. Since the agent can follow the strategy of exercising the option at threshold $X_P^*(\theta)$, it must be that $V_A^b(X,\theta)\ge\left(\frac{X}{X_P^*(\theta)}\right)^{\beta}(\theta X_P^*(\theta)-I+b)$. By contradiction, suppose that there exists point $\hat{X}<X_P^*(\theta)$ at which it is optimal for the agent to exercise the option. Then, since $V_A^b(X,\theta)\ge\left(\frac{X}{X_P^*(\theta)}\right)^{\beta}(\theta X_P^*(\theta)-I+b)$, it must be that

$$V_A^b(X,\theta)\ge\left(\frac{X}{X_P^*(\theta)}\right)^{\beta}(\theta X_P^*(\theta)-I+b)\le\left(\frac{X}{\hat{X}}\right)^{\beta}\left(\theta\hat{X}-I+b\right),$$

which is a contradiction since the right-hand side is strictly increasing in $\hat{X}\in(0,X_P^*(\theta))$. Thus, the exercise threshold is in the range $X\ge X_P^*(\theta)$. Eq. (80) must be solved subject to the value-matching and smooth-pasting conditions, we obtain

$$V_A^b(X,\theta)=\begin{cases}BX^{\gamma_+}+\left(\frac{X}{X_P^*(\theta)}\right)^{\beta}(\theta X_P^*(\theta)-I+b), & \text{if } X\le X_P^*(\theta),\\ A_1X^{\gamma_-}+A_2X^{\gamma_+}+\frac{\lambda\theta}{r+\lambda-\alpha}X-\frac{\lambda(I-b)}{r+\lambda}, & \text{if } X\in\left[X_P^*(\theta),\tilde{X}_A(\theta)\right],\\ \theta X-I+b, & \text{if } X\ge\tilde{X}_A(\theta),\end{cases}$$

where $\gamma^+>1$ and $\gamma^-<0$ are the roots of $\frac{1}{2}\sigma^2\gamma(\gamma-1)+\alpha\gamma-r-\lambda=0$, constants A_1, A_2, and B are given by

$$A_1 = \frac{X_P^*(\theta)^{-\gamma_-}}{\gamma_+ - \gamma_-}\left(\theta X_P^*(\theta)\left(\frac{(r-\alpha)(\gamma_+-1)}{r+\lambda-\alpha} - \beta + 1\right) - (I-b)\left(\frac{r\gamma_+}{r+\lambda} - \beta\right)\right),$$

$$A_2 = \frac{\tilde{X}_A(\theta)^{-\gamma_+}}{\gamma_+ - \gamma_-}\left(\theta \tilde{X}_A(\theta)\frac{(r-\alpha)(1-\gamma_-)}{r+\lambda-\alpha} + (I-b)\frac{r\gamma_-}{r+\lambda}\right),$$

$$B = A_2 - \frac{X_P^*(\theta)^{-\gamma_+}}{\gamma_+ - \gamma_-}\left(\theta X_P^*(\theta)\left(\beta - \gamma_- - \frac{\lambda(1-\gamma_-)}{r+\lambda-\alpha}\right) - (I-b)\left(\beta - \frac{r\gamma_-}{r+\lambda}\right)\right),$$

and the optimal exercise threshold $\tilde{X}_A(\theta)$ satisfies

$$\theta \tilde{X}_A(\theta)\frac{(r-\alpha)(\gamma_+-1)}{r+\lambda-\alpha} - (I-b)\frac{r\gamma_+}{r+\lambda} = \left(\frac{\tilde{X}_A(\theta)}{X_P^*(\theta)}\right)^{\gamma_-}\left(\begin{array}{c} I\left(\frac{\beta}{\beta-1}\frac{(r-\alpha)(\gamma_+-1)}{r+\lambda-\alpha} - \frac{r\gamma_+}{r+\lambda}\right) \\ -b\left(\beta - \frac{r\gamma_+}{r+\lambda}\right)\end{array}\right). \quad (81)$$

The left-hand side is strictly increasing in $\tilde{X}_A(\theta)$. Let us see that the right-hand side is strictly decreasing in $\tilde{X}_A(\theta)$. Since $\gamma_- < 0$ and $b < 0$, it is sufficient to show that $\frac{\beta}{\beta-1}\frac{(r-\alpha)(\gamma_+-1)}{r+\lambda-\alpha} > \frac{r\gamma_+}{r+\lambda}$ and $\beta > \frac{r\gamma_+}{r+\lambda}$. Using the definition of γ_+, $\frac{\gamma_+}{\gamma_+-1}\frac{r+\lambda-\alpha}{r+\lambda} = 1 + \frac{\sigma^2\gamma_+}{2(r+\lambda)}$. Let us show that $\frac{\gamma_+}{r+\lambda}$ is strictly decreasing in λ:

$$\frac{\sigma^2\gamma_+}{r+\lambda} = \sqrt{\left(\rho\left(\alpha - \frac{\sigma^2}{2}\right)\right)^2 + 2\rho\sigma^2} - \rho\left(\alpha - \frac{\sigma^2}{2}\right),$$

where $\rho \equiv 1/(r+\lambda)$. Differentiating with respect to ρ and using $\sqrt{1+x} < 1 + \frac{x}{2}$ for $x > 0$, we obtain

$$\frac{-\rho\left(\alpha - \frac{\sigma^2}{2}\right)^2\left(\sqrt{1 + \frac{2\sigma^2}{\rho\left(\alpha - \frac{\sigma^2}{2}\right)^2}} - 1\right) + \sigma^2}{2\sqrt{\left(\rho\left(\alpha - \frac{\sigma^2}{2}\right)\right)^2 + 2\rho\sigma^2}} > 0.$$

Therefore, $\frac{\gamma_+}{r+\lambda}$ is indeed strictly decreasing in λ. Hence, $\frac{\gamma_+}{\gamma_+-1}\frac{r+\lambda-\alpha}{r+\lambda} < \frac{\beta}{\beta-1}\frac{r-\lambda}{r}$ and $\frac{r\gamma_+}{r+\lambda} < \frac{r\beta}{r}$. The latter inequality proves $\beta > \frac{r\gamma_+}{r+\lambda}$. Multiplying the former inequality by $\frac{r(\gamma_+-1)}{r+\lambda-\alpha}$ proves $\frac{\beta}{\beta-1}\frac{(r-\alpha)(\gamma_+-1)}{r+\lambda-\alpha} > \frac{r\gamma_+}{r+\lambda}$. Hence, the right-hand side of (81) is strictly decreasing in $\tilde{X}_A(\theta)$. Therefore, there exists a unique $\tilde{X}_A(\theta)$ that solves (81). To show that $\tilde{X}_A(\theta) < X_A^*(\theta)$, suppose by contradiction that $\tilde{X}_A(\theta) > X_A^*(\theta)$.[22] Since waiting is optimal in $X < \tilde{X}_A(\theta)$, $V_A^b(X,\theta) > \theta X - I + b\ \forall X \in \left[X_A^*(\theta), \tilde{X}_A(\theta)\right)$. Since $V_A^*(X,\theta) = \theta X - I + b\ \forall X \geq X_A^*(\theta)$, we obtain $V_A^b(X,\theta) > V_A^*(X,\theta)\ \forall X \in \left[X_A^*(\theta), \tilde{X}_A(\theta)\right)$, which is a contradiction with $V_A^*(X,\theta)$ being the highest possible value function to the agent across all exercise policies. Thus, $\tilde{X}_A(\theta) < X_A^*(\theta)$.

Next, we show that the strategy profile stated in the proposition constitutes an equilibrium in the communication game, where $\tilde{X}_A(\theta)$ is defined by (81). The IC condition for the agent with $\theta : \tilde{X}_A(\theta) \leq \check{X}$ is satisfied, since it leads to the option being exercised at threshold $\tilde{X}_A(\theta)$, which is the optimal strategy for the agent in the constrained delegation problem, as shown above. The IC condition for the agent with

[22] It is easy to see that $X_A^*(\theta)$ does not solve (81), so $\tilde{X}_A(\theta) \neq X_A^*(\theta)$.

$\theta : \tilde{X}_A(\theta) > \check{X}$ is satisfied, since threshold $\check{X}$ is the highest threshold at which the agent can get the option to be exercised, given the strategy of the principal, and $\check{X}$ dominates any threshold below it by monotonicity of the agent's payoff.

Finally, it remains to derive threshold $\check{X}$, at which it is optimal for the principal to exercise without waiting for the agent's recommendation. Let $V_P^b(X, \theta, Y^*)$ denote the value function to the principal prior to the arrival of the news, conditional on the type of the agent being θ and conditional on the option being exercised at threshold $Y^* \geq X$ prior to the arrival of the news. $\tilde{V}_P^b(X, \theta, Y^*)$ solves

$$(r+\lambda)\tilde{V}_P^b(X,\theta,Y^*) = \alpha X \frac{\partial \tilde{V}_P^b(X,\theta,Y^*)}{\partial X} + \frac{1}{2}\sigma^2 X^2 \frac{\partial^2 \tilde{V}_P^b(X,\theta,Y^*)}{\partial X^2} + \lambda V_P^a(X,\theta), \tag{82}$$

where

$$V_P^a(X,\theta) = \begin{cases} \left(\frac{X}{X_P^*(\theta)}\right)^\beta (\theta X_P^*(\theta) - I), & \text{if } X \leq X_P^*(\theta) \\ \theta X - I, & \text{if } X \geq X_P^*(\theta) \end{cases} \tag{83}$$

is the value of the option to the principal after the arrival of the news. Eq. (82) is solved subject to the value-matching condition $\tilde{V}_P^b(Y^*, \theta, Y^*) = \theta Y^* - I$. Let $V_P^b\left(X, \hat{\theta}, \check{X}\right)$ be the principal's value function prior to the arrival of the news, given current state X, posterior belief that θ is uniform over $\left[\underline{\theta}, \hat{\theta}\right]$, if the principal waits for the agent's recommendation to exercise until threshold $\check{X}$:

$$V_P^b\left(X,\hat{\theta},\check{X}\right) = \int_{\underline{\theta}}^{\hat{\theta}} \tilde{V}_P^b\left(X,\theta,\min\left\{\tilde{X}_A(\theta),\check{X}\right\}\right)\frac{1}{\hat{\theta}-\underline{\theta}}d\theta.$$

Differentiating with respect to $\check{X}$, we obtain the first-order condition that determines $\check{X}$:

$$\int_{\underline{\theta}}^{\tilde{X}_A^{-1}(\check{X})} \frac{\partial \tilde{V}_P^b\left(X,\theta,\check{X}\right)}{\partial \check{X}} d\theta = 0. \tag{84}$$

■

Proof of Proposition 9. **Proof of part 1.** We start by showing that under the specified restrictions on b, θ_L and θ_H indeed exist. First, if $b \in (0, \frac{\mathbb{E}[\theta]-\underline{\theta}}{\mathbb{E}[\theta]}I)$, then $\underline{\theta}\frac{I}{I-b} < \mathbb{E}[\theta]$ and $\bar{\theta}\frac{I}{I-b} > \bar{\theta}$. Since $\mathbb{E}\left[\tilde{\theta}|\tilde{\theta} \geq \theta\right]$ is continuous in θ, the equation $\theta\frac{I}{I-b} = \mathbb{E}\left[\tilde{\theta}|\tilde{\theta} \geq \theta\right]$ has at least one solution, and all solutions are strictly below $\bar{\theta}$. Hence, $\theta_H < \bar{\theta}$. Second, if $b \in (-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}I, 0)$, then $\bar{\theta}\frac{I}{I-b} > \mathbb{E}[\theta]$ and $\underline{\theta}\frac{I}{I-b} < \underline{\theta}$. Since $\mathbb{E}\left[\tilde{\theta}|\tilde{\theta} \leq \theta\right]$ is continuous in θ, the equation $\theta\frac{I}{I-b} = \mathbb{E}\left[\tilde{\theta}|\tilde{\theta} \leq \theta\right]$ has at least one solution. By Assumption 2, the solution is unique and $\theta_L > \underline{\theta}$.

We next show that our problem satisfies the conditions of Proposition 1 in Amador and Bagwell (2013), once we introduce a change in process from $X(t)$ to $P(X(t)) \equiv X(t)^{-\beta+1}$. Since $\beta > 1$, $P(X)$ is strictly decreasing in X with $\lim_{X\to 0} P(X) = \infty$ and $\lim_{X\to\infty} P(X) = 0$. Thus, there is a one-to-one correspondence between X and P, so any threshold-exercise direct mechanism $\left\{\hat{X}(\theta), \theta \in \Theta\right\}$ with upper thresholds on process $X(t)$ can be equivalently written as a threshold-exercise direct mechanism $\left\{\hat{P}(\theta), \theta \in \Theta\right\}$ with lower thresholds on process $P(X(t))$, with $\hat{P}(\theta) = \hat{X}(\theta)^{-\beta+1}$. The payoffs of the principal and the agent, $\tilde{U}_P\left(\hat{P},\theta\right)$ and $\tilde{U}_A\left(\hat{P},\theta\right)$, divided by $X(0)^{\beta}$, can be written as:

$$\begin{aligned}\tilde{U}_P\left(\hat{P},\theta\right) &= \frac{\theta\hat{P}^{\frac{1}{1-\beta}}-I}{\hat{P}^{\frac{\beta}{1-\beta}}}=\theta\hat{P}-I\hat{P}^{\frac{\beta}{\beta-1}}\\ \tilde{U}_A\left(\hat{P},\theta\right) &= \frac{\theta\hat{P}^{\frac{1}{1-\beta}}-(I-b)}{\hat{P}^{\frac{\beta}{1-\beta}}}=\theta\hat{P}-(I-b)\,\hat{P}^{\frac{\beta}{\beta-1}}\end{aligned}$$

Thus, the optimal mechanism problem (similar to (4)–(5), but for a general distribution) is a special case of the problem in Amador and Bagwell (2013) without money burning, where $\gamma=\theta$, $\pi=\hat{P}$, $\omega\left(\theta,\hat{P}\right)=\tilde{U}_P\left(\hat{P},\theta\right)$, and $b\left(\hat{P}\right)=-(I-b)\,\hat{P}^{\frac{\beta}{\beta-1}}$. It is easy to check that the conditions of Assumption 1 in their paper hold for any $b<I$. By analogy with Amador and Bagwell (2013), define κ as

$$\kappa\equiv\frac{\frac{\partial^2\tilde{U}_P\left(\hat{P},\theta\right)}{\partial\hat{P}^2}}{b''\left(\hat{P}\right)}=\frac{I}{I-b}.$$

We next verify that the conditions of Proposition 1 in Amador and Bagwell (2013) hold for these functions. Since $X_A^*(\theta)=\frac{\beta}{\beta-1}\frac{I-b}{\theta}$, the optimal agent's threshold in terms of $P(t)$ is $P_A^*(\theta)=\left(\frac{\beta}{\beta-1}\frac{I-b}{\theta}\right)^{1-\beta}$. Then, $\frac{\partial\tilde{U}_P\left(P_A^*(\theta),\tilde{\theta}\right)}{\partial\hat{P}}=\tilde{\theta}-\frac{I}{I-b}\theta$.

Condition (c1). Since $\frac{\partial\tilde{U}_P(P_A^*(\theta),\theta)}{\partial\hat{P}}=-\frac{b}{I-b}\theta$, (c1) is satisfied if and only if $\Phi(\theta)+\frac{b}{I}\theta\phi(\theta)$ is non-decreasing for all $\theta\in\left[\underline{\theta},\bar{\theta}\right]$.

Condition (c2). This condition is relevant in the case $b\in(0,\frac{\mathbb{E}[\theta]-\underline{\theta}}{\mathbb{E}[\theta]}I)$, in which case $\theta_H<\bar{\theta}$. In this case, we need to verify

$$(\theta-\theta_H)\frac{I}{I-b}\geq\int_{\theta}^{\bar{\theta}}\left(\tilde{\theta}-\frac{I}{I-b}\theta_H\right)\frac{\phi\left(\tilde{\theta}\right)}{1-\Phi(\theta)}d\tilde{\theta}\ \forall\theta\in\left[\theta_H,\bar{\theta}\right]$$

with equality at θ_H. Rearranging the terms, this inequality is equivalent to $\theta\frac{I}{I-b}\geq\mathbb{E}\left[\tilde{\theta}|\tilde{\theta}\geq\theta\right]$. By definition of θ_H, this condition indeed holds as equality at θ_H. Furthermore, since θ_H is the highest $\theta\in\Theta$ at which $\theta\frac{I}{I-b}=\mathbb{E}\left[\tilde{\theta}|\tilde{\theta}\geq\theta\right]$ and because $\bar{\theta}\frac{I}{I-b}>\bar{\theta}$, this inequality holds strictly at any $\theta\in\left(\theta_H,\bar{\theta}\right]$.

Condition (c2'). This condition is relevant in the case $b<0$. In this case, we need to verify $\frac{\partial\tilde{U}_P}{\partial\hat{P}}\left(P_A^*\left(\bar{\theta}\right),\bar{\theta}\right)\geq 0$, which is equivalent to $\bar{\theta}-\frac{I}{I-b}\bar{\theta}\geq 0$. It is satisfied if and only if $b<0$.

Condition (c3). This condition is relevant in the case $b\in(-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}I,0)$, in which case $\theta_L>\underline{\theta}$. In this case, we need to verify

$$(\theta-\theta_L)\frac{I}{I-b}\leq\int_{\underline{\theta}}^{\theta}\left(\tilde{\theta}-\frac{I}{I-b}\theta_L\right)\frac{\phi\left(\tilde{\theta}\right)}{\Phi(\theta)}d\tilde{\theta}\ \forall\theta\in\left[\underline{\theta},\theta_L\right]$$

with equality at θ_L. Rearranging the terms, this inequality is equivalent to $\theta\frac{I}{I-b}\leq\mathbb{E}\left[\tilde{\theta}|\tilde{\theta}\leq\theta\right]$. By definition of θ_L, it holds as equality at θ_L. Furthermore, since θ_L is the unique $\theta\in\Theta$ at which $\theta\frac{I}{I-b}=\mathbb{E}\left[\tilde{\theta}|\tilde{\theta}\leq\theta\right]$ and because $\underline{\theta}\frac{I}{I-b}<\underline{\theta}$, this inequality holds strictly at any $\theta\in(\underline{\theta},\theta_L]$.

Condition (c3'). This condition is relevant in the case $b>0$. We need to verify $\frac{\partial\tilde{U}_P}{\partial\hat{P}}\left(P_A^*\left(\underline{\theta}\right),\underline{\theta}\right)\leq 0$, which is equivalent to $\underline{\theta}-\frac{I}{I-b}\underline{\theta}\leq 0$. It is satisfied if $b\in(0,I)$ and hence is satisfied if $b\in(0,\frac{\mathbb{E}[\theta]-\underline{\theta}}{\mathbb{E}[\theta]}I)$.

Applying Proposition 1 in Amador and Bagwell (2013), we conclude that the optimal threshold-exercise decision rule is $\hat{X}(\theta) = X_A^*(\min\{\theta, \theta_H\})$ if $b \in (0, \frac{\mathbb{E}[\theta]-\underline{\theta}}{\mathbb{E}[\theta]}I)$, and $\hat{X}(\theta) = X_A^*(\max\{\theta, \theta_L\})$, if $b \in (-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}I, 0)$.

Proof of Part 2. Given that the principal plays the strategy stated in the proposition, the strategy of any type θ of the agent is incentive-compatible. Indeed, for any type $\theta \geq \theta_L$, exercise occurs at his most preferred time, so no type $\theta \geq \theta_L$ benefits from a deviation. Any type $\theta < \theta_L$ does not benefit from a deviation either because the agent would lose from inducing the principal to exercise earlier, and inducing exercise later than threshold $X_A^*(\theta_L)$ is not feasible given that the principal never exercises later than $X_A^*(\theta_L)$ under her strategy. Next, let us verify the optimality of the principal's strategy. We need to check that the principal has incentives to exercise the option immediately when the agent sends message $m = 1$ (the ex-post IC constraint), and not to exercise the option before getting message $m = 1$ (the ex-ante IC constraint). The ex-post IC constraint follows from the fact that the principal learns the agent's type θ if the agent sends message $m = 1$ at first-passage time of any threshold between $X_A^*(\bar{\theta})$ and $X_A^*(\theta_L)$, and realizes that it is already too late to exercise ($X_P^*(\theta) < X_A^*(\theta)$), and thus does not benefit from delaying exercise even further. If the agent sends a message to exercise when $X(t)$ hits $X_A^*(\theta_L) = \frac{\beta}{\beta-1}\frac{I-b}{\theta_L}$, the principal infers that $\theta \leq \theta_L$ and that she will not learn any additional information by waiting more. Given the belief that $\theta \in [\underline{\theta}, \theta_L]$, the optimal exercise threshold for the principal is given by $\frac{\beta}{\beta-1}\frac{I}{\mathbb{E}[\theta|\theta\leq\theta_L]}$, which equals $\frac{\beta}{\beta-1}\frac{I-b}{\theta_L}$ by the definition of θ_L. Hence, the ex-post IC constraint is satisfied. Finally, consider the ex-ante IC constraint.

Proof that the principal's ex-ante IC constraint is satisfied. Let $\tilde{V}_p\left(X, \hat{\theta}, \theta_L\right)$ be the expected value to the principal from following the strategy of waiting for the agent's recommendation $m = 1$ until $X(t)$ hits $X_A^*(\theta_L)$ for the first time and exercising at threshold $X_A^*(\theta_L)$ regardless of the agent's recommendation, where X is the current value of $X(t)$ and the principal believes that θ is distributed over $\left[\underline{\theta}, \hat{\theta}\right]$, $\hat{\theta} \geq \theta_L$ with p.d.f. $\phi(\theta)/\Phi(\hat{\theta})$:

$$\tilde{V}_p\left(X, \hat{\theta}, \theta_L\right) = X^\beta \left(\int_{\theta_L}^{\hat{\theta}} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta}\frac{\phi(\theta)}{\Phi(\hat{\theta})}d\theta + \int_{\underline{\theta}}^{\theta_L} \frac{\theta X_A^*(\theta_L) - I}{X_A^*(\theta_L)^\beta}\frac{\phi(\theta)}{\Phi(\hat{\theta})}d\theta\right).$$

Because the principal's belief is that $\theta \in \left[\underline{\theta}, \hat{\theta}\right]$, the current value of $X(t)$ satisfies $X(t) \leq X_A^*(\hat{\theta})$. Hence, the ex-ante IC constraint requires $\tilde{V}_p\left(X, \hat{\theta}, \theta_L\right) \geq X\mathbb{E}\left[\theta|\theta \leq \hat{\theta}\right] - I$ for any $\hat{\theta} > \theta_L$ and $X \leq X_A^*\left(\hat{\theta}\right)$, or, equivalently,

$$\int_{\theta_L}^{\hat{\theta}} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta}\frac{\phi(\theta)}{\Phi(\hat{\theta})}d\theta + \int_{\underline{\theta}}^{\theta_L} \frac{\theta X_A^*(\theta_L) - I}{X_A^*(\theta_L)^\beta}\frac{\phi(\theta)}{\Phi(\hat{\theta})}d\theta \geq \frac{X\mathbb{E}\left[\theta|\theta \leq \hat{\theta}\right] - I}{X^\beta}. \tag{85}$$

The right-hand side of (85) is an inverted U-shaped function that reaches its maximum at $X_{\max} = \frac{\beta}{\beta-1}\frac{I}{\mathbb{E}\left[\theta|\theta\leq\hat{\theta}\right]}$. Since equation $\mathbb{E}\left[\tilde{\theta}|\tilde{\theta} \leq \theta\right] = \frac{I}{I-b}\theta$ has a unique solution $\theta_L \in \Theta$ and since $\mathbb{E}[\theta] < \frac{I}{I-b}\bar{\theta}$ for $b \in (-\frac{\bar{\theta}-\mathbb{E}[\theta]}{\mathbb{E}[\theta]}I, 0)$, then $\mathbb{E}\left[\theta|\theta \leq \hat{\theta}\right] \leq \frac{I}{I-b}\hat{\theta}$ for any $\hat{\theta} \geq \theta_L$. Therefore, $X_{\max} = \frac{\beta}{\beta-1}\frac{I}{\mathbb{E}\left[\theta|\theta\leq\hat{\theta}\right]} \geq \frac{\beta}{\beta-1}\frac{I-b}{\hat{\theta}} = X_A^*\left(\hat{\theta}\right)$, and hence the right-hand side of (85) is strictly increasing in X over $X \leq X_A^*\left(\hat{\theta}\right)$. Hence, the ex-ante IC constraint (85) is satisfied for any $X \leq X_A^*\left(\hat{\theta}\right)$ if and only if it is satisfied at $X = X_A^*\left(\hat{\theta}\right)$. Finally, suppose that (85) is violated at $X = X_A^*\left(\hat{\theta}\right)$ for some $\hat{\theta} > \theta_L$. However, this implies that threshold schedule $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\max\{\theta,\theta_L\}}$ is dominated by $\hat{X}(\theta) = \frac{\beta}{\beta-1}\frac{I-b}{\max\{\theta,\hat{\theta}\}}$, which contradicts part 1 of the proposition. Indeed, violation of (85) means

$$\int_{\theta_L}^{\hat{\theta}} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta}\frac{\phi(\theta)}{\Phi(\hat{\theta})}d\theta + \int_{\underline{\theta}}^{\theta_L} \frac{\theta X_A^*(\theta_L) - I}{X_A^*(\theta_L)^\beta}\frac{\phi(\theta)}{\Phi(\hat{\theta})}d\theta < \frac{X_A^*\left(\hat{\theta}\right)\mathbb{E}\left[\theta|\theta \leq \hat{\theta}\right] - I}{X_A^*\left(\hat{\theta}\right)^\beta}. \tag{86}$$

The principal's expected utility under the optimal contract in part 1 of Proposition 9, divided by $X(0)^\beta$, is

$$\int_{\underline{\theta}}^{\theta_L} \frac{\theta X_A^*(\theta_L) - I}{X_A^*(\theta_L)^\beta}\phi(\theta)\,d\theta + \int_{\theta_L}^{\bar{\theta}} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta}\phi(\theta)\,d\theta. \tag{87}$$

Consider a modified contract with θ_L replaced by $\hat{\theta}$. The principal's expected utility under this modified contract, divided by $X(0)^\beta$, is

$$\int_{\underline{\theta}}^{\hat{\theta}} \frac{\theta X_A^*\left(\hat{\theta}\right) - I}{X_A^*\left(\hat{\theta}\right)^\beta}\phi(\theta)\,d\theta + \int_{\hat{\theta}}^{\bar{\theta}} \frac{\theta X_A^*(\theta) - I}{X_A^*(\theta)^\beta}\phi(\theta)\,d\theta. \tag{88}$$

Rearranging the terms, it is straightforward to see that (87)<(88) is equivalent to (86). Hence, (86) implies that the contract in part 1 of Proposition 9 is dominated by another interval delegation contract, which is a contradiction. Thus, the ex-ante IC constraint is indeed satisfied, which completes the proof of Part 2.

Proof of Part 3. First, consider the agent's strategy. By Assumption 1, sending a message $m = 1$ prior to delegation of authority does not change the principal's belief and hence her strategy. Thus, the agent cannot induce exercise before he is given authority. After the agent is given authority at threshold $X_A^*(\theta_H)$, the agent's optimal exercise strategy is to exercise at $X_A^*(\theta)$ if $\theta < \theta_H$, and immediately if $\theta \geq \theta_H$. Second, consider the principal's strategy. Since delegation at threshold $X_A^*(\theta_H)$ implements the optimal mechanism from part 1 of the proposition, the same argument as in the proof of Proposition 7 applies to show that delegating at threshold $X_A^*(\theta_H)$ is the optimal strategy for the principal. ■

C. Robustness

In this section, we show the robustness of the results to several versions of the model.

C.1 Simple compensation contracts

A reasonable question is whether simple compensation contracts, such as paying a fixed amount for exercise (if $b < 0$) or for the lack of exercise (if $b > 0$), can solve the problem and thus make the analysis less relevant. We show that this is not the case. Specifically, we allow the principal to offer the agent the following payment scheme. If the agent is biased towards late exercise ($b < 0$), the principal can promise the agent a lump-sum payment z that he will receive as soon as the option is exercised. A higher payment decreases the conflict of interest and speeds up option exercise. For example, if $z = \frac{-b}{2}$, the agent's and the principal's interests are aligned because each of them receives $\theta X - I + \frac{b}{2}$ upon exercise. However, a higher payment is also more expensive for the principal. Because of that, as the next result shows, it is always optimal for the principal to offer $z^* < \frac{-b}{2}$, and hence the conflict of interest will remain. Moreover, if the agent's bias is sufficiently small, the optimal payment is in fact zero.

Similarly, if the agent is biased towards early exercise ($b > 0$), the principal can promise the agent a flow of payments $\hat{z}dt$ up to the moment when the option is exercised. Higher $\hat{z}$ aligns the interests of the players but is expensive for the principal. The next result shows that if the initial value of the state process is sufficiently small, the optimal $\hat{z}$ is again zero. In numerical analysis, we also show that similarly to the late exercise bias case, the optimal payment is smaller than the payment that would eliminate the conflict of interest.

Proposition C.1. *Suppose $b < 0$ and the principal can promise the agent a payment $z \geq 0$ upon exercise. Then the optimal z is always strictly smaller than $\frac{-b}{2}$ and equals zero if $b > \frac{-I}{\beta-1}$. Suppose $b > 0$ and the principal can promise the agent a flow of payments $\hat{z}dt \geq 0$ up to the moment of option exercise. Then the optimal $\hat{z}$ equals zero if $X(0)$ is sufficiently small.*

Thus, allowing simple compensation contracts often does not change the problem at all, and at most leads to an identical problem with a different bias b. We conclude that the problem and implications of our paper are robust to allowing simple compensation contracts.

Proof of Proposition C.1. First, consider $b < 0$. The payoffs of the principal and the agent upon exercise are given by $\theta X - I - z$ and $\theta X - I + b + z$, respectively. Hence, the problem is equivalent to the problem of the basic model with $I' \equiv I + z$ and $b' = 2z + b$. The interests of the principal and the agent become aligned if $b' = 0$, i.e., if $z = \frac{-b}{2}$. Note that it is never optimal to have $z > 0$ if $b' < -I'$: in this case, the equilibrium will feature uninformed exercise and hence would give the principal the same expected utility as if he did not make any payments. Similarly, it is never optimal to have $b' > 0$. Hence, we can restrict attention to $b' \in [-I, 0]$. Then, the most informative equilibrium of the communication game features continuous exercise, and according to (10), the principal's expected utility as a function of z is

$$V(z) = \frac{X(0)^\beta}{\beta+1}\left(\frac{\beta}{\beta-1}(I'-b')\right)^{-\beta}\frac{I'-\beta b'}{\beta-1} = \frac{X(0)^\beta}{\beta^2-1}\left(\frac{\beta}{\beta-1}\right)^{-\beta}(I-b-z)^{-\beta}(I-\beta b+z(1-2\beta)).$$

Note that $V'(z) > 0 \Leftrightarrow z < z^*$, where $z^* = \frac{-(I-b)(2\beta-1)+\beta I-\beta^2 b}{(\beta-1)(2\beta-1)}$. It is easy to show that $z^* > 0 \Leftrightarrow b < \frac{-I}{\beta-1}$ and that $z^* < -\frac{b}{2} \Leftrightarrow (\beta-1)(b-2I) < 0$, which holds for any $b < 0$. This completes the proof of the first statement.

Next, consider $b > 0$. If the principal makes flow payoffs $\hat{z}dt$ before exercise, then upon exercise the agent loses $\frac{\hat{z}}{r}$, which is the present value of continuation payments at that moment. Thus, the principal's and agent's effective payoffs upon exercise are $\theta X(t) - I + \frac{\hat{z}}{r}$ and $\theta X(t) - I + b - \frac{\hat{z}}{r}$, respectively. Hence, we can consider the communication game with $I' = I - \frac{\hat{z}}{r}$ and $b' = b - 2\frac{\hat{z}}{r}$. The interests of the principal and the agent become aligned if $b = 2\frac{z}{r}$, i.e., if $z = \frac{rb}{2}$. Similarly to the case $b < 0$, it is never optimal to have $\hat{z} > 0$ if $b' \geq I'$ or $b' < 0$, and hence we can restrict attention to $b' \in [0, I')$. Denoting $\tilde{z} \equiv \frac{\hat{z}}{r}$ and using (16), the payoff of the principal at the initial date is

$$V(\tilde{z}) = -\tilde{z} + \frac{1-\omega}{1-\omega^{\beta+1}} \left(\frac{X(0)}{Y(\omega, \tilde{z})} \right)^{\beta} \left(\frac{1}{2}(1+\omega) Y(\omega, \tilde{z}) - I + \tilde{z} \right),$$

where by (8), $Y(\omega, \tilde{z}) = \frac{\left(1-\omega^{\beta}\right)(I-b+\tilde{z})}{\omega(1-\omega^{\beta-1})}$. By (72), the most informative equilibrium of this game is characterized by $\omega = \frac{1}{\frac{\beta}{\beta-1}\frac{1-\omega^{\beta-1}}{1-\omega^{\beta}}\frac{2(I-\tilde{z})}{I-b+\tilde{z}}-1}$. If $X(0) \to 0$, $V'(\tilde{z}) \to -1$, and hence $\tilde{z} = 0$ is optimal, which completes the proof. ■

C.2 Model with different discount rates

In our basic setup, the conflict of interest between the agent and the principal is modeled by the agent's bias b. Our results are similar in an alternative setup, where the conflict of interest arises because the agent and the principal have different discount rates. This section presents the summary of this analysis, and the full analysis is available from the authors upon request.

Suppose that the agent's discount rate is r_A, the principal's discount rate is r_P, and both players' payoff from exercise at time t is $\theta X(t) - I$. Similar to the basic model, we can define β_A and β_P, where β_i is the positive root of the quadratic equation $\frac{1}{2}\sigma^2 \beta_i(\beta_i - 1) + \alpha\beta_i - r_i = 0$.

The case where the principal is more impatient than the agent ($r_P > r_A$, or equivalently, $\beta_P > \beta_A$) is similar to the case $b < 0$ in the basic model. We show that if $\underline{\theta} = 0$, then as long as $\beta_A > \frac{2\beta_P}{1+\beta_P}$, there exists an equilibrium with continuous exercise in which exercise occurs at the agent's most preferred threshold $\frac{\beta_A}{\beta_A - 1}\frac{I}{\theta}$. If $\underline{\theta} > 0$, the equilibrium features continuous exercise up to a cutoff. The case where the agent is more impatient than the principal ($r_P < r_A$) is similar to the case $b > 0$ in the basic model. We show that the equilibrium with continuous exercise does not exist and derive the analog of Proposition 2. Specifically, in the most informative stationary equilibrium, exercise is unbiased given the principal's information. This equilibrium is characterized by $\tilde{\omega}^* < 1$, which is the unique solution of

$$\frac{\left(1-\omega^{\beta_A}\right) I}{\omega\left(1-\omega^{\beta_A - 1}\right)} = \frac{\beta_P}{\beta_P - 1}\frac{2I}{\omega + 1}.$$

In addition, for any $\omega \in [\underline{\tilde{\omega}}, \tilde{\omega}^*)$, where $0 < \underline{\tilde{\omega}} < \tilde{\omega}^*$, there is a unique ω-equilibrium where exercise happens with delay.

C.3 Put option

So far, we have assumed that the decision problem is over the timing of exercise of a call option, such as the decision of when to invest. In this section, we show that if the decision problem is over the timing of exercise of a put option, such as the decision of when to liquidate a project, the analysis and economic insights are similar. The nature of the option, call or put, is irrelevant for the results. What matters is the asymmetric nature of time: Time moves forward and thereby creates a one-sided commitment device for the principal to follow the agent's recommendations.

Consider the model of Section 2 with the following change. The exercise of the option leads to the payoffs $\theta I - X(t)$ and $\theta(I+b) - X(t)$ for the principal and the agent, respectively. As before, θ is a random draw from a uniform distribution on $[\underline{\theta}, 1]$ and is privately learned by the agent at the initial date. If $\underline{\theta} = 0$, the model exhibits stationarity. For example, if the decision represents shutting down a project, $I\theta$ corresponds to the salvage value of the project, $b\theta$ represents the agent's private cost (if $b < 0$) or benefit (if $b > 0$) of liquidating the project, and $X(t)$ corresponds to the present value of the cash flows from keeping the project afloat. The solution of this model follows the same structure as the solution of the model with the call option. We summarize our findings below, and the full analysis is available from the authors upon request.

Suppose that we start with a high enough $X(0)$, so that immediate exercise does not happen. At the beginning of Section B of the Online Appendix, we show that if θ were known, the optimal exercise policy of each player would be given by a lower trigger on $X(t)$: $X_P^{**}(\theta) = \frac{\delta}{\delta+1} I\theta$, $X_A^{**}(\theta) = \frac{\delta}{\delta+1}(b+I)\theta$, where $-\delta$ is the negative root of the quadratic equation that defined β. If $b > 0$, then $X_A^{**}(\theta) > X_P^{**}(\theta)$, i.e., the agent's preferred exercise policy is to exercise earlier than the principal. Similarly, if $b < 0$, the agent is biased towards late exercise.

Suppose that $\underline{\theta} = 0$ and consider the communication game like the one in Section 4. If $b \in (-\frac{I}{2}, 0)$, there is an equilibrium with full information revelation: The agent recommends to wait as long as $X(t)$ exceeds her preferred exercise threshold $X_A^{**}(\theta)$ and recommends exercise at the first moment when $X(t)$ hits $X_A^{**}(\theta)$. Upon getting the recommendation to exercise, the principal realizes it is too late and finds it optimal to exercise immediately. Prior to that, the principal prefers to wait because the value of learning θ exceeds the cost of delay. If $b > 0$, this equilibrium does not exist, and all stationary equilibria are of the form $\{(\omega, 1), (\omega^2, \omega), ...\}$, where type $\theta \in (\omega^n, \omega^{n-1})$ recommends exercise at threshold $\omega^{n-1} Y_{put}(\omega)$, where $Y_{put}(\omega) = \frac{\omega - \omega^{\gamma+1}}{1-\omega^{\gamma+1}}(I+b)$.

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