**A Dynamic Noisy Rational Expectation Model with Higher Order Beliefs**[1]

**Bruce Grundy and Qi Zeng**[2]

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**Abstract**

We study a dynamic noisy rational expectation model with higher order beliefs. Our baseline model includes three types of agents, Informed (I), Uninformed (U) and Partially informed (M). The information structure is hierarchical: The information set of the agents I includes that of the agents M, which include that of the agents U. A crucial condition for the existence of higher-order beliefs is the existence of enough linear independent (public) signals to the least informed agents U. Otherwise the updating of their beliefs collapse to lower order beliefs. Higher order beliefs affect the expected excess returns. For example, the agents U's high expectation about the agents M's belief about the fundamentals leads U believing that M is over-buying, and thus U will over-sell. The agents M and I, being more informed and thus knowing the agents U being making a mistake, want indeed to buy. The net effect from this shock is small but negative on the stock price. Higher order beliefs also dramatically enrich the trading volume study. The set of the buyers and sellers are different depending on different types of shocks.

**Keywords: Higher order belief, noisy rational expectation, trading volume, expected returns**

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**Preliminary Draft and Comments Welcome**

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[2] Grundy is from University of Melbourne, Carlton, VIC, Australia, Email: bruceg@unimelb.edu.au. Zeng is from University of Melbourne, Carlton, VIC, Australia, Email: qzeng@unimelb.edu.au.

## 1. Introduction

When agents in a competitive economy have heterogeneous information, the equilibrium asset price is a function of not only the agents' beliefs about the underlying fundamentals of the economy, but also the beliefs about other agents' beliefs, or higher order beliefs. The potential state space is of infinite order. This is the famous “infinite regress” problem pointed out by Townsend (1978). Various studies have shown the effects of higher order beliefs[3]. Yet with the few exceptions discussed later, there are not many methods dealing with the infinite regress problem.

In this paper we generalize the hierarchical information structure normally used in the asymmetric information models to incorporate higher order beliefs. In the standard asymmetric information models (Townsend, 1981, Grossman and Stiglitz, 1980), the information structure across the agents is hierarchical. More specifically, uninformed agents do not know the true fundamental states of the economy. They infer the underlying fundamental states from public signals such as dividends and prices. Informed agents have the full information including all the underlying fundamentals as well as the signals received by the uninformed agents, if any. Thus the informed agents do not need to form beliefs about the uninformed agents' beliefs, because they know the uninformed agents' beliefs.

In our model, the information structure across agents is still hierarchical in the sense that the information set of one type of agents is a subset of the other type of agents. However, the agents with more information do not necessarily have all the information about the economy. In the baseline model we study in the main text, there are three types of agents I, U

[3] For example, heterogeneous information based model has been used in rational explanation of crashes (Grundy and McNichols, 1989), technique analysis (Brown and Jennings, 1989), trading (say, all information based models in micro-structure, among others), volatility (Veronesi, 1999), and more recently price persistency (Allen, Morris and Shin, 2006; Makarov and Rytchkov, 2012). The recent rekindled interest in higher order beliefs is that higher order beliefs might be the key to understand the price persistence /reversal, and many other asset pricing anomalies associated with the volatile features of returns compared with the fundamentals (Bacchetta and van Wincoop, AER, 2006; JMCB, 2008). But technique challenges prevent us from doing much along this direction (Banerjee, Kaniel, and Kremer, 2009; Banerjee and Kremer, 2010; Banerjee, 2011).

and M. The agents I are the informed agents who know every information in the economy. The agents U are uninformed agents who only know the public signals. The agents M have more information than agents U in the sense that they have some private information that agents U do not know. However, the agents M are not fully informed. For example, the agents I know about the underlying fundamentals while the agents M only receive signals about the fundamentals.

Suppose that the underlying state of the economy is characterized by two fundamental state variables $(F, \Theta)$[4]. There is a stock whose payoff is driven by at least of the fundamental states. The Agents I, being fully informed, know these two variables. However, neither the agents M nor the agents U observe them directly. Instead the agents U observe only public signals, such as the stock price $P$, the stock dividend $D$, and maybe an additional public signal $S_U$. The agents M, being more informed, observe $(P, D, S_U)$ plus a private signal $S_M$. Given this information, the agents M form the beliefs about the underlying state, denoted as $(\hat{F}^M, \widehat{\Theta}^M)$. The optimal demand of the agents M is a function of these beliefs. Thus the equilibrium price of the asset is generally a function of both $(F, \Theta)$ and $(\hat{F}^M, \widehat{\Theta}^M)$.

The agents U also have to form beliefs about the future payoffs of holding the stock conditional on their (public) signals. Since the equilibrium price includes both the fundamentals and the beliefs of the agents M, the agents U have to form the beliefs not only about $(F, \Theta)$, denoted as $(\hat{F}^U, \widehat{\Theta}^U)$, but also about $(\hat{F}^M, \widehat{\Theta}^M)$. As a result, the second-order belief, denoted as $(\bar{\bar{F}}, \bar{\bar{\Theta}})$, arises naturally in the agents U's optimal demand. The equilibrium price, resulting from the market clearing condition

[4] The reason we need two states here is to prevent the full-revealing equilibrium. If there is only one state variable and the informed agents know about it, in equilibrium the price will fully reveal this (Grossman and Stiglitz, 1980).

in the stock market, is a function of not only the fundamentals $(F, \Theta)$, first-order beliefs of the agents M and U about the fundamentals, but also the second-order beliefs of the agents U. Furthermore, there are no other higher order of beliefs because the agents M (and the agents I) know all the information of the agents U, thus know about $(\hat{F}^U, \hat{\Theta}^U, \bar{\bar{F}}, \bar{\bar{\Theta}})$. As a result, the equilibrium price is only a function of $(F, \Theta, \hat{F}^M, \hat{\Theta}^M, \hat{F}^U, \hat{\Theta}^U, \bar{\bar{F}}, \bar{\bar{\Theta}})$.

One subtlety here is the number of signals that agents U must have. Looking at the above structure, it seems that $S_U$ is not needed since there are already two public signals $(P, D)$ available to all the agents. Indeed in the current asymmetric information models, one normally assumes that the uninformed agents only know about these. However in our setting the existence of more public signals is crucial. The agents U need to form expectations on more than two state variables through Bayesian learning. Thus if there are only two linear independent (not perfectly correlated) signals, the higher order beliefs may be reduced to first order beliefs because they are not linear independent to the other first order beliefs.

From this discussion, we can easily generalize the above model to include more types of partially informed agents. For example one can assume three types of partially informed agents with hierarchical information structure in addition to the full informed agents. We study the requirements and the resulting equilibrium of such a setup in details in the discussion section. In this case, the third order belief arises naturally from similar consideration as above.

Introducing a third type of agents into the standard asymmetric information model enriches our study of information based asset pricing and liquidity models enormously. The excess return is driven by not only fundamentals and different investors' beliefs of these fundamentals, but also the less informed investors' beliefs about the more informed investors' beliefs. A shock to any of these fundamental and derived states affects the prices. For example, consider the case when the forecasting error of the second-order belief, denoted as $\Delta$, defined as the difference between the agents U's belief about the agents M's belief, $\bar{\bar{F}}$, and the agents M's actual belief, $\hat{F}^M$, is high. Even though the agents U do not realize this over-estimating mistake directly, it does lead the agents U to believe that the agents M make over-estimating mistake. This is because when the agents U look at the agents M's forecasting error about the fundamentals: $\hat{F}^M - F$, they believe that it is $E^U(\hat{F}^M - F) = \bar{\bar{F}} - \hat{F}^U$. This can be written as the following:

$$\bar{\bar{F}} - \hat{F}^U = (\bar{\bar{F}} - \hat{F}^M) + (\hat{F}^M - F) - (\hat{F}^U - F).$$

The first term is the forecasting error of the second-order belief, $\Delta$. The last two terms are the forecasting error by the two partially informed agents U and M. Thus when $\Delta$ is high, agents U believe that the agents M make a mistake by over-estimating the fundamentals, *ceteris paribus.* They believe that the agents M will over-buy, and thus the agents U will over-sell. Of course, both the agents I and M know that the uninformed agents U make a mistake by over-estimating the agents M's expectation, thus they are more likely to buy because the agents U over-sell. The net effect is a small but negative effect on the asset price.

Introducing a third type of agents also leads to much richer structure to the trading volume study. In models with only two (types of) agents, a study on the trading volume is essentially a study on one agent's portfolio problem, because of market clearing condition. In our setting, even though it is still the case that the net purchasing amount of one type of the agents equal to the net selling amount of the other two types of the agents, the actual buyers and sellers are different for different shocks. As discussed above, if there is a positive shock on the agents U's the second-order belief, the agents M (and to a lessor extent, the agents I) are the net buyer while the agents U are the net seller, *ceteris paribus*. However, if there is a shock to the agents M's first order belief about the fundamentals, agents M will still be the net buyer (because they think this is a good buy), but agents I will be the net seller (because they know agents M are likely to make a mistake). More interestingly, agents U will also be a net seller, because when agents M over-estimate, it is more likely that agents U's second order belief is also high.

Our paper is closely linked to two streams of literature. One is a continuous effort to come up with an effective method to study the effect of higher order beliefs under heterogeneous information. Different methods have been proposed. For example, one can assume that the information will be revealed after some time, say one period (Lucas, 1975, Townsend, 1983, Singleton, 1987). One can also assume that payouts are iid, and agents only receive signals about finite horizon. Or one can guess that the price functional is an ARMA process and work on the frequency domain (Sargent, 1991, Kasa, 2000). Finally one can smartly design the model so that higher order beliefs are reduced to first-order beliefs (He and Wang, 1995. However, see Makarov and Rytchkov, 2012, for a counter argument). Our contribution to this literature is to generalize the hierarchical information structure so that we can study the higher order beliefs in the spirit of the large asymmetric information literature discussed next.

The other stream of literature is the hierarchical information structure. This includes almost all the financial literature with asymmetric information. However, just as the solution suggested by Townsend (1981) to resolve the infinite regress problem, all the models assume that there are two types of agents, informed and uninformed (Other classic works include Grossman, 1976; Grossman and Stiglitz, 1980, etc., and market microstructure models such as Glosten and Milgrom, 1985; Kyle 1985, etc. ). The idea to generalize the hierarchical information structure to incorporate more than two types of agents has been suggested in Wang (1993). However, to the best of our knowledge, we are the first to build such a model. Our contribution to the literature is to provide a rigorous framework to study the effects of higher order beliefs under asymmetric information assumption. Our model exhibits the rich modelling structure and empirical implications in the asset pricing and trading literature by introducing higher order beliefs.

The paper is organized as follows. In the next section, we provide an illustrative example of two-period three-date model to show the basic intuition on the setup of the model. We then present the basic setup of our benchmark model. Then we solve the equilibrium prices in the next section. After obtaining the solutions, we study in details the effect of different orders of beliefs on the expected excess returns, individual stock demand and tradings. This is followed by a generalization of the model to include long-lived agents with intertemporal consumption as well as several generalizations of the benchmark model. Finally we conclude. All the proofs are in the appendix.

## 2. An Illustrative Model

We start by considering a variation of the model studied in Vayanos and Wang (2011, 2013), in which they survey both the theoretical and empirical studies on liquidity. We use this setup to illustrate the intuition and the conditions needed for the existence of higher order beliefs under hierarchical information structure. Furthermore we can see the effects of higher order beliefs on measures related to liquidity and expected returns. Since the setup is very similar to those in both Vayanos and Wang (2011, 2013) and the formal model later in this paper, we state the results without proof in this section.

There are three dates $t = 0, 1, 2$ and two securities, one riskless bond and one risky stock. The risk free rate is exogenously given and normalized to zero. The risky stock pays a random dividend $D$ at time $t = 2$. The dividend $D$ consists of two parts:

$$D = \Pi + \epsilon_D. \tag{1}$$

The aggregate supply of the shares $\Theta$ is also randomly distributed[5]. Ex ante, all the agents believe that $(\Pi, \epsilon_D, \Theta)$ are uncorrelated with each other, and normally distributed with mean zero and variances $\left(\sigma_\pi^2, \sigma_D^2, \sigma_\theta^2\right)$ respectively.

There are a continuum of agents distributed uniformly on $[0,1]$. All the agents have the same negative exponential utility function with risk aversion coefficient $\gamma > 0$. At time $t = 1$, a proportion $\omega \in [0,1]$ of the agents learn both $\Pi$ and $\Theta$, thus become "informed"'. It is straight forward to show that the equilibrium price at time $t = 1$,

$$P_1 = b(\Pi - c\Theta), \tag{2}$$

where

$$b = \frac{\omega^2\sigma_\Pi^2 + \omega(\sigma_D^2 + \sigma_\Pi^2)\gamma^2\sigma_D^2\sigma_\Theta^2}{\omega^2\sigma_\Pi^2 + (\sigma_D^2 + \omega\sigma_\Pi^2)\gamma^2\sigma_D^2\sigma_\Theta^2} \tag{3}$$

$$c = \frac{\gamma\sigma_D^2}{\omega}.$$

When $\omega \to 0$, $b \to 0$ and $\mathrm{P}_1 \to -\gamma\sigma_\mathrm{D}^2\Theta$, since there are no "informed" agents in the economy. And when $\omega \to 1$, $b \to 1$ and $\mathrm{P}_1 \to \Pi - \gamma\sigma_\mathrm{D}^2\Theta$, since all the agents in the economy are "informed".

Under this price function, the informed and uninformed agents hold the following number of shares of the stock each:

$$\theta_1^j = \frac{E_1^j(D) - P_1}{\gamma\sigma_j^2(D)}, \tag{4}$$

[5] One can instead assume a private investment opportunity available only to one agent, who is then informed. This is the approach adopted in Vayanos and Wang (2011, 2013).

where $j = i, u$ indicates the expectations of the informed and uninformed agents. More specifically,

$$E_1^i(D) = \Pi, \quad \sigma_i^2(D) = \sigma_D^2$$

$$E_1^u(D) = \frac{\sigma_\Pi^2}{\sigma_\Pi^2 + c^2\sigma_\Theta^2}(\Pi - c\Theta), \quad \sigma_u^2(D) = \frac{\sigma_D^2\sigma_\Pi^2 + c^2\sigma_\Theta^2(\sigma_D^2 + \sigma_\Pi^2)}{\sigma_\Pi^2 + c^2\sigma_\Theta^2}.$$

At time 0, all the agents are identical, they need to take into account that with probability $\omega \in [0,1]$ they might become informed, and $1 - \omega$ that they might not. Denote the price at time 0 as $P_0$. Vayanos and Wang (2011, 2013) define two liquidity measures using the above stylized example. The first one is the negative covariance between the dollar returns over the two periods, and the second one is the beta of regressing the dollar returns on the trading volume:

$$\lambda_1 \equiv -cov(D_2 - P_1, P_1 - P_0), \tag{5}$$

$$\lambda_2 \equiv \frac{cov\left(P_1 - P_0, \omega\left(\theta_1^i - \theta_0\right)\right)}{var\left(\omega\left(\theta_1^i - \theta_0\right)\right)}. \tag{6}$$

They argue that the first measure includes both a permanent component and a transitory component, and the second measure is mainly a transitory measure. In the current setup, the two measures are:

$$\lambda_1 = -b(1-b)\sigma_\Pi^2 + b^2c^2\sigma_\Theta^2,$$

$$\lambda_2 = \frac{b(\sigma_\Pi^2 + c^2\sigma_\Theta^2)}{\omega\left((1-b)\sigma_\Pi^2 - bc^2\sigma_\Theta^2\right)}.$$

The expected return of the stock over the two periods are: $\mathrm{E}_0(\mathrm{R}) = \frac{\bar{\Pi}}{\mathrm{P}_0}$. We will study the implications of higher order beliefs on these three measures, the two liquidity measures and the expected returns, later after solving the full dynamic model.

For now we want to use this simple setup to discuss the necessary conditions for the existence of higher order beliefs under hierarchical information structure. Our main focus is at time t=1. Under the current setup with just two types of agents, the informed and the uninformed, we can rewrite the price as:

$$P_1 = b\left(1 + \frac{c^2\sigma_\Theta^2}{\sigma_\Pi^2}\right)E_1^u(D).$$

In other words, the equilibrium price is a linear function of the expected payoffs of the uninformed agents.

To introduce higher order beliefs under the hierarchical information structure, we can introduce a type of "partially informed" agents, indexed by m, in addition to the informed and uninformed agents. For example, we can assume that even though these agents don't know about Π, they can receive a private signal, $s_m = \Pi + \epsilon_m$, about $\Pi$, where $\epsilon_m$ is a normal random error. As a result, agents m form their beliefs about the payoff, $E_1^m(D)$, from public signal, equilibrium price $P_1$ and the private signal $s_m$. The equilibrium price is then a (linear) function of $E_1^m(D)$. Under the hierarchical information structure, we can assume that the informed agents know $s_m$ as well, but the uninformed agents don't. So $s_m$ is a private signal for both i and m agents.

This leads to the second order beliefs for the uninformed agents u. Since the signal available to the uninformed agents is only the public signal $P_1$, the uninformed agents use this signal (and the prior) to form their beliefs about the payoff $D$, $E_1^u(D)$, and the beliefs about the beliefs of agents m, $E_1^u[E_1^m(D)]$. As a result, the equilibrium price will be a function of the second order belief $E_1^u[E_1^m(D)]$ as well.

However, this is not enough to generate a generic second order belief. If the only signal available to the uninformed agents is $P_1$ and these agents use it to form two beliefs,

$E_1^u(D)$ and $E_1^u[E_1^m(D)]$, the two beliefs cannot be linear independent to each other. For the two beliefs to be linear independent, we may give the uninformed agents another signal $s_u$, which can be a function of $\Pi$ or $E_1^m(D)$ (e.g. $s_m$). Under the hierarchical information structure, we assume that $s_u$ is known by all the agents, thus it is a public signal. As a result, the equilibrium price $P_1$ will be a function of the first order beliefs $E_1^m(D), E_1^u(D)$ and the second order belief $E_1^u[E_1^m(D)]$. Looking back, the uninformed agents are not totally "uninformed" since they also receive additional signals in addition to the prices. The crucial part is that the agents' information sets are hierarchical.

To briefly summarize, we need a partially informed agent m so that the equilibrium price is a function of $E_1^m(D)$. This in turn gives rise to the second order beliefs of uninformed agents from the information in the price. In order to support a generic second order belief, however, the uninformed agents also need other sources of information in addition to the price. We will not expand this example further. Instead we go on to study the formal dynamic version of the model in the following.

## 3. Model Setup

Consider an economy with two securities, one risk free bond and one risky stock. Assume that the bond is in perfect elastic supply and its return is constant $R = 1 + r$, where $r \geq 0$. The stock pays a dividend $D_t$:

$$D_t = F_t + \epsilon_{D,t}, \tag{7}$$

where the mean $F_t$ follows:

$$F_t = a_F F_{t-1} + \epsilon_{F,t}, a_F \in (0,1). \tag{8}$$

Time is discrete from negative infinity to positive infinity[6] and we will search for a stationary solution. As is often the case with differential information setup, the agents in the economy live for only two periods and thus only needs to solve for a static maximization problem. The purpose of this assumption is such that we can concentrate the discussion on the information and higher order beliefs. Later on, we will discuss the generalization to infinitely lived agents situation.

As such, we assume that the economy is spanned by overlapping generations of short-lived agents. Each period, there are a continuum of agents born, and a continuum of agents die. Both the new and old agents are uniformly distributed on the interval [0, 1]. Define

$$Q_{t+1} \equiv P_{t+1} + D_{t+1} - RP_t, \tag{9}$$

as the excess share returns. Each new born agent j maximizes a one-period constant absolute risk aversion (CARA) utility function,

$$E_{j,t}[-e^{-\gamma W_{j,t+1}}],$$

where

$$W_{j,t+1} = W_{j,t}R + \theta_{j,t+1}Q_{t+1}, \tag{10}$$

and $\theta_{j,t+1}$ is the shares held by the agent j born at time $t$ . If $Q_{t+1}$ is normally distributed, as will be the case in this paper, it follows:

$$\theta_{j,t+1} = \frac{E_{j,t}(Q_{t+1})}{\gamma\, \sigma_{j,t}^2(Q_{t+1})}. \tag{11}$$

[6] As it will be clear from the discussion in the following, one can easily generalize the model to the continuous time situation. This is solely because of the hierarchical information structure. The generalization cannot be easily done within models with heterogeneous information because of the infinite regress problem.

Here again for simplicity we assume that all agents have the same risk aversion coefficients $\gamma$. As a result, the sole heterogeneity of the agents comes from information difference because there is no wealth effect for CARA utility function.

For future reference, we denote the total supply of stocks to the new born generation at time t as $\Theta_t$, which can be time varying and stochastic to avoid full revealing equilibrium. The implication of this assumption is that the incremental supply of the shares is the difference of $\Theta_t$ and $\Theta_{t-1}$.[7]

3.1. **Information Structure**

We assume that there are three types of agents within each generation, informed, uninformed and partially informed. The informed agents, denoted as I, have some private information that only agents I know, and know about all the other information in the economy. The uninformed agents, denoted as U, have no private information about the mean payoff beyond the public signals. The partially informed agents, denoted as M, have some information that the uninformed agents U do not know, but agents I know. In summary, the information set of the informed agents I include the information set of the partially informed agents M, which include the information set of the uninformed agents U.

Agents I and U are the two types of the agents that are typically studied in the asymmetric information models in the literature. The unique feature in our model is the appearance of the partially informed agents M, whose information set is between the sets of I and U. We now specify the information set of each type of agents.

The uninformed agents U receive a signal about the mean payoff at any time t:

[7] Allen and Shin (2006) uses similar assumption to simplify the model.

$$S_{U,t} = F_t + \epsilon_{U,t}. \quad (12)$$

This signal is known to all the agents in the economy, and thus a public signal. The information set of the agents U, $I_{U,t}$, at any time includes the history of all the public signals up until time t, $I_{U,t} = \{P_\tau, D_\tau, S_{U,\tau}\}_{\tau \le t}$. We will discuss the importance of the additional public signal $S_{U,t}$ in a moment.

The partially informed agents M have all the information of the agents U. Additionally they also receive a private signal about $F_t$ that the agents U do not know:

$$S_{M,t} = F_t + \epsilon_{M,t}. \quad (13)$$

The information set of the agents M, $I_{M,t}$, at any time includes the history of all the public signals and the private signals up until time t, $I_{M,t} = \{P_\tau, D_\tau, S_{U,\tau}, S_{M,\tau}\}_{\tau \le t}$.

The informed agents I know about all the private and public signals in the economy, including the actual mean payoffs, $F_t$. The agents' information set at any time t is thus $I_{I,t} = \{P_\tau, D_\tau, F_\tau, S_{U,\tau}, S_{M,\tau}\}_{\tau \le t}$.[8] It is clear from the above setup that the hierarchical information structure holds throughout, $I_{U,t} \subset I_{M,t} \subset I_{I,t}, \forall t$.

Finally we assume that the aggregate supply of the risky stock for new born populations follow:

$$\Theta_t = a_\Theta \Theta_{t-1} + \epsilon_{\Theta,t}, a_\Theta \in (0,1) \quad (14)$$

In the following, we assume that the noise terms, $\epsilon_t \equiv \left(\epsilon_{F,t}, \epsilon_{\Theta,t}, \epsilon_{D,t}, \epsilon_{U,t}, \epsilon_{M,t}\right)'$, are i.i.d normally distributed with mean zero, and the variances-covariance matrix, $\Sigma$, defined as $\Sigma \equiv diag(\sigma_F^2, \sigma_\Theta^2, \sigma_D^2, \sigma_U^2, \sigma_M^2)$ being a diagonal matrix. The normal distribution assumption is

[8] Here we assume that the new born informed and partially informed agents inherent the information of their respective old generations. This assumption simplifies the state updating process for these agents. The key point is to keep the hierarchical information structure. Again this assumption is not crucial for the long-lived agent situation that we will discuss later.

standard in the models with CARA utility function. The assumption of uncorrelated noise is a simplification assumption. And finally we denote the proportion of the three types of agents as $\omega_I, \omega_U, \omega_M \geq 0$, and $\omega_I + \omega_U + \omega_M = 1$.

## 4. Solutions

### 4.1. Benchmark Cases

We first discuss briefly several benchmark cases that the literature has studied so far.

#### 4.1.1. Symmetric Information

The first benchmark case is when all agents receive the same information, namely all the information is public. Under our setup, this is the case when there are only one type of agents. We differentiate between the full information and partial information structure. Full information structure is the case in which all agents are informed, like agents I. Partial information structure is the case in which all the agents are of type U[9].

**Proposition 1. (Full Information Equilibrium)**

*Under the full information assumption, the equilibrium price of the stock is given by:*

$$P_t \; = \; p_1^* \, F_t \; - \; p_\Theta^* \Theta_t, \tag{15}$$

*where the constants* $p_1^*, p_\Theta^*$ *satisfy*:

$$p_1^* = \frac{a_F}{R - a_F} > 0 \tag{16}$$

*and*

$$0 \; = \gamma \, \sigma_\Theta^2 \, p_\Theta^{*2} \; - \; (R \; - \; a_\Theta) p_\Theta^* \; + \gamma \, [(1 + p_1^*)^2 \, \sigma_F^2 \; + \sigma_D^2]. \tag{17}$$

[9] In the symmetric information case, the only difference between all U agents and all M agents is that there is one more (public) signal in the latter case.

*Specifically when the aggregate supply of the stock is fixed at $\Theta_t = \Theta$, namely $a_\Theta = 1$, $\sigma_\Theta^2 = 0$ in equation (14), the constant $p_\Theta^*$ becomes*:

$$p_\Theta^* = \frac{\gamma[\sigma_F^2(1+p_1^*)^2 + \sigma_D^2]}{r} > 0. \tag{18}$$

That the price is a linear function of the state variable, the mean payoff $F_t$ is the result of CARA utility function and the normal distributional assumption. In this full-information economy, every agent holds the same amount of shares, $\Theta$, in equilibrium. There is no trading across time, while the stock prices change to fully account for the shocks in the economy.

Under the symmetric partial information structure, all agents observe $D_t$ and $S_{U,t}$. They infer $\mathrm{F_t}$ from the history $\{D_\tau, S_{U,\tau}\}, \tau \leq t$. We assume that agents know $\Theta_t$ each period or it is a constant through the time. The price is then a linear function of the belief about $\mathrm{F}_t$. :

**Proposition 2. (Symmetric Partial Revealing Equilibrium)**

*Under symmetric partial information structure, there exists a stationary linear equilibrium. In this equilibrium, agents update their beliefs about $\mathrm{F_t}$ according to*:

$$\hat{F}_{\mathrm{t+1|t+1}} = \mathrm{a_F}\hat{F}_{\mathrm{t|t}} + \frac{1}{\Gamma_\mathrm{D} + \Gamma_\mathrm{U} + \Gamma_\mathrm{F}}\left[\Gamma_\mathrm{D}\left(\mathrm{D_{t+1}} - \mathrm{a_F}\hat{\mathrm{F}}_{\mathrm{t|t}}\right) + \Gamma_\mathrm{U}\left(\mathrm{S_{U,t+1}} - \mathrm{a_F}\,\hat{\mathrm{F}}_{t|t}\right)\right]. \tag{19}$$

*The variance of current belief about $F_t, \Omega \equiv E_t\left(F_t - \left(\hat{F}_{t|t}\right)\right)^2$, satisfies*:

$$(\Gamma_\mathrm{D} + \Gamma_\mathrm{U})\mathrm{a_F^2}\,\Omega^2 + [(\Gamma_\mathrm{D} + \Gamma_\mathrm{U})\sigma_\mathrm{F}^2 + 1 - \mathrm{a_F^2}]\Omega - \sigma_\mathrm{F}^2 = 0. \tag{20}$$

*where*

$$\Gamma_\mathrm{D} \equiv \frac{1}{\sigma_\mathrm{D}^2}, \qquad \Gamma_\mathrm{U} \equiv \frac{1}{\sigma_\mathrm{U}^2}, \qquad \Gamma_\mathrm{F} \equiv \frac{1}{\mathrm{a_F^2}\Omega + \sigma_\mathrm{F}^2}. \tag{21}$$

*The equilibrium price satisfies*:

$$P_t = p_1^* \hat{F}_{t|t} - p_\Theta^{**} \Theta \tag{22}$$

*where* $p_1^*$ *is defined in equation (16) of Proposition 1, and*

$$\begin{aligned} &\gamma(\Gamma_D + \Gamma_U + \Gamma_F)^2 \sigma_\Theta^2 p_\Theta^{**2} - (R - a_\Theta)(\Gamma_D + \Gamma_U + \Gamma_F)^2 p_\Theta^{**} + \\ &\gamma[a_F^2\big((\Gamma_D + \Gamma_U)(1 + p_1^*) + \Gamma_F\big)^2 \Omega + \big((\Gamma_D + \Gamma_U)(1 + p_1^*) + \Gamma_F\big)^2 \sigma_F^2 + \\ &\big((1 + p_1^*)\Gamma_D + \Gamma_U + \Gamma_F\big)^2 \sigma_D^2 + p_1^{*2}\Gamma_U] = 0 \end{aligned} \tag{23}$$

There are two interesting links between the two symmetric information equilibriums. The first is when the public signal $S_{U,t}$ fully reveals $F_t$, namely $\sigma_U^2 = 0$ or $\Gamma_U \to \infty$. Then the belief $\hat{F}_{t|t} = F_t$ is the true underlying state and $\Omega = 0$. The equilibrium becomes the symmetric full information case in the Proposition 1, $p_\Theta^{**} = p_\Theta^*$.

The other is when the public signal $D_t$ is also fully revealing, $\sigma_D^2 = 0$ or $\Gamma_D \to \infty$. Again $\Omega = 0$ in this case, because the belief of $F_t$ is the same as $F_t$. And the equation about $p_\Theta^*$ and $p_\Theta^{**}$ becomes:

$$0 = \gamma\sigma_\Theta^2\, p_\Theta^{*2} - (R - a_\Theta)p_\Theta^* + \gamma(1 + p_1^*)^2\sigma_F^2.$$

This is the same fully revealing equilibrium with $\sigma_D^2 = 0$ in Proposition 1.

### 4.1.2. Asymmetric Information with Fully Informed Agents

The asymmetric information models have been studied extensively in the literature. Our setup in this case is mostly similar to that in Wang (1993, 1994). There are just two types of agents I and U in this situation. The informed agents I know about $\mathrm{F_t}$, while uninformed agents only observe public signals such as prices and dividends. For now we assume there is no public signal $s_B$, the following result is straight forward:

**Proposition 3. (Asymmetric Information Equilibrium)**

*Under the assumption that type I agents know about the state of the economy,* $\{F_t, \Theta_t, P_t, D_t\}$*, and type U agents only know about public signals* $\{P_t, D_t\}$*, there exists an equilibrium with the prices as:*

$$P_t = p_1 F_t + p_2 \hat{F}_t - p_\Theta \Theta_t \tag{24}$$
$$= (p_1 + p_2) F_t + p_2 \delta_{F,t} - p_\Theta \Theta_t,$$

*where* $\hat{F}_t$ *is the belief of type U's agents about the underlying state* $F_t$*, and* $\delta_{F,t} \equiv \hat{F}_t - F_t$ *is the error of the beliefs of the U agents.*

The above equilibrium price has zero constant terms. This is consistent with previous symmetric results since the constant terms in those situations are linear functions of aggregate supply $\Theta$. Here $\Theta_t$ is a state variable, so the constant term is zero. In Makarov and Rytchkov (2012) the constant term is not zero because in their set up, the mean incremental supply is one. Another difference between the above solution and that in Wang (1993, 1994) is that the coefficient $p_1$ is a constant not necessarily the same as $p_1^*$ defined in the symmetric case. This is consistent with Makarov and Rytchkov (2012).

The asymmetric information assumption is the basic assumption underlying almost all of the heterogeneous information models in asset pricing. Namely informed agents know about the true states of the economy while uninformed agents don't. In reality, it is very hard to claim any agents know about the true states of the economy. A more realistic assumption is that the informed agents know more about the states than uninformed agents. As we will see in the next section, this simple change of assumption leads to higher order beliefs, even under two agent types cases.

### 4.2. **Asymmetric Information with Partially Informed Agents**

When some agents have only partial information about the economy, the more informed agents' average beliefs about the underlying states become state variables. Furthermore, the less informed agents not only have to take expectation about the true underlying states, but also have to take expectations about the average beliefs of the more informed investors. Thus higher order beliefs arise naturally in this set up. In the following we show heuristically the conditions for the higher order beliefs, which we have alluded to in the illustrative two-period example.

The fundamental states of the economy include the mean payoffs $\mathrm{F}_t$ and aggregate share supply $\Theta_t$, and the informed agents I know about these. The resulting equilibrium price is a function of $\mathrm{F}_t, \Theta_t$. Furthermore, there are derived states, the expectations of both agents U and M. In general the price function is of the following format, $P_t = L(F_t, \Theta_t, \hat{F}_t^U, \hat{\Theta}_t^U, \hat{F}_t^M, \hat{\Theta}_t^M, \bar{\bar{F}}_t, \bar{\bar{\Theta}}_t)$ . Here $\bar{\bar{F}}_t \equiv E^U(\hat{F}_t^M)$, is the U agents' expectation of M agents' expectations about the state $\mathrm{F}_t$, $\bar{\bar{\Theta}}_t \equiv E^U(\hat{\Theta}_t^M)$ is the U agents' expectations of M agents' expectations about the state $\Theta_t$, and $L(\cdot)$ is the linear functional operator:

$$L\big(F_t, \Theta_t, \hat{F}_t^U, \hat{\Theta}_t^U, \hat{F}_t^M, \hat{\Theta}_t^M, \bar{\bar{F}}_t, \bar{\bar{\Theta}}_t\big) = p_F F_t - p_\Theta \Theta_t + p_{\hat{F}^U} \hat{F}_t^U + p_{\hat{\Theta}^U} \hat{\Theta}_t^U + p_{\hat{F}^M} \hat{F}_t^M + p_{\hat{\Theta}^M} \hat{\Theta}_t^M + p_{\bar{\bar{F}}} \bar{\bar{F}}_t + p_{\bar{\bar{\Theta}}} \bar{\bar{\Theta}}_t.$$

The linear functional format is from the CARA utility and the normal distributions assumption[10].

[10] We do not consider the nonlinear price function`s and uniqueness of the equilibrium in this paper.

The existence of the first-order beliefs of the two agents U and M come from that the fact that neither agents are fully informed. More importantly, U agents' second-order beliefs also become a state variable, since from U's point of view, the states of the economy include both fundamentals as well as M's beliefs. And these are all the state variables that determine the economy, since the hierarchical information structure dictates that the agents I know about all the information that U and M have, and the agents M know about all the information that the agents U have. Thus there is no "infinite regress" problem. This situation is the same as that in the standard asymmetric information models, except that there are higher order beliefs involved.

### 4.2.1. Redundant State Variables

Even though in general the price functional is a linear function of all the state variables, not all of them are linear independent. Let us start with the more informed agents M. For the agents M, observing the price $P_t$ is equivalent to observing $\Pi_t^M \equiv p_F F_t - p_\Theta \Theta_t$, where $p_F, p_\Theta$ are constant coefficients in the price functional. This is because the agents M know all of the agents U's information and thus beliefs. They are just not certain about the underlying fundamental states $(F_t, \Theta_t)$. Taking expectations with respect to M's information set, it follows that:

$$p_F F_t - p_\Theta \Theta_t = p_F \hat{F}_t^M - p_\Theta \hat{\Theta}_t^M. \tag{25}$$

Rewriting the above expression results in the following two equivalent expressions:

$$p_F \delta_{F,t}^M = p_\Theta \delta_{\Theta,t}^M \tag{26}$$

and

$$\hat{\Theta}_t^M = \Theta_t + \frac{p_F}{p_\Theta} \delta_{F,t}^M, \tag{27}$$

where

$$\delta_{F,t}^{M} \equiv \hat{F}_t^M - F_t, \qquad \delta_{\Theta,t}^{M} \equiv \hat{\Theta}_t^M - \Theta_t, \tag{28}$$

represent the forecasting errors of the M agents.

The above discussion implies that in the price function, there is no need to have $\widehat{\Theta}_t^M$ as an independent state variable since it is a linear function of $\{F_t, \Theta_t, \hat{F}_t^M\}$, or $\{\Theta_t, \delta_{F,t}^M\}$. Of course this result is already known in the literature (see, for example, the arguments given in Wang (1993, 1994)). In the standard asymmetric information models, the informed agents know about all the state variables including the uninformed agents' expectations. And one of the uninformed agents' expectations about the fundamental variables is redundant. In our setup with higher order beliefs, the partially informed agents are not fully informed, thus one of their expectations is also redundant.

Furthermore, the above discussion also implies that the uninformed agents U know about the above relationship. Thus when U agents take expectations of the above relationship, it follows that:

$$\bar{\bar{\Theta}}_t = \hat{\Theta}_t^U + \frac{p_F}{p_\Theta}\left(\bar{\bar{F}}_t - \hat{F}_t^U\right) = \hat{\Theta}_t^U + \frac{p_F}{p_\Theta}\hat{\delta}_{F,t}^U, \tag{29}$$

where

$$\hat{\delta}_{F,t}^U \equiv \bar{\bar{F}}_t - \hat{F}_t^U \tag{30}$$

represents the expectation of the agents U on the forecasting errors of the agents M. In other words, $\bar{\bar{\Theta}}_t$ is not an independent state variable either, since it is a linear function of $\{\hat{\Theta}_t^U,\ \hat{\delta}_t^U\}$, or $(\hat{\Theta}_t^U, \hat{F}_t^U, \bar{\bar{F}}_t\}$. As a result, we can write the price function as the form, $P_t = L\left(F_t, \Theta_t, \hat{F}_t^M, \hat{F}_t^U, \hat{\Theta}_t^U, \bar{\bar{F}}_t\right)$.

The number of derived state variables can be further reduced. When the agents U observe the prices, they are equivalent to observing $\Pi_t^U = p_F F_t - p_\Theta \Theta_t + p_M \hat{F}_t^M$, where $p_M$

is the constant coefficient in the price functional. Namely agents U do not know the components of $\Pi_t^U$, but they know the combination of them. Taking expectations with respect to U's information set, it follows that

$$p_F F_t - p_\Theta \Theta_t + p_M \hat{F}_t^M = p_F \hat{F}_t^U - p_\Theta \hat{\Theta}_t^U + p_M \bar{\bar{F}}_t. \tag{31}$$

Or

$$\hat{\Theta}_t^U = \Theta_t + \frac{p_F}{p_\Theta} \delta_{F,t}^U + \frac{p_M}{p_\Theta} \Delta_t^U, \tag{32}$$

where

$$\delta_{F,t}^U \equiv \hat{F}_t^U - F_t, \qquad \Delta_t^U \equiv \bar{\bar{F}}_t - \hat{F}_t^M. \tag{33}$$

Note the difference between $\hat{\delta}_{F,t}^U$ and $\Delta_t^U$. $\hat{\delta}_{F,t}^U$ is the U agents' expectation about the M agents' forecasting error, which, from U agents point of view, is known. $\Delta_t^U$ is the forecasting error of U agents' expectation of M agents' expectation about $F_t$, which the U agents don't know about. Indeed using the fact that $\hat{\delta}_{F,t}^U = \bar{\bar{F}}_t - \hat{F}_t^U = \left(\bar{\bar{F}}_t - \hat{F}_t^M\right) + (\hat{F}_t^M - \hat{F}_t^U)$, it follows that:

$$\hat{\delta}_{F,t}^U = \Delta_t^U + \delta_{F,t}^M - \delta_{F,t}^U. \tag{34}$$

The discussion implies that $\hat{\Theta}_t^U$ is not an independent state variable since it is a function of $\{F_t, \Theta_t, \hat{F}_t^U, \hat{F}_t^M, \bar{\bar{F}}_t\}$. Combining the above discussion together, we can see that only one of $\hat{\Theta}_t^U, \bar{\bar{F}}_t$ is an independent state variable. We choose $\bar{\bar{F}}_t$ as the one used in the price function to indicate the higher order beliefs of the U agents.

One might argue that the above linear relationship among $(\hat{\Theta}_t^U, \bar{\bar{F}}_t, \bar{\bar{\Theta}}_t)$ seems to indicate that the higher order beliefs can be reduced to first-order beliefs, just like He and Wang (1995). Using $\bar{\bar{\mathrm{F}}}_{\mathrm{t}}$ seems to disguise this effect: The discussion about $\bar{\bar{F}}_t$ is essentially the discussion on $\hat{\Theta}_t$. As we will argue in the discussions section later, this is not true in general. One can easily generalize the basic intuition embedded in the above set up to beliefs of order higher than two. The resulting simplification cannot reduce all the higher order beliefs to the first-order. In other words, the higher order beliefs in this set up is generic. What we have shown in the current set up is just the simplest possible set up of incorporating higher order beliefs. Much of the intuition holds qualitatively for general higher order beliefs.

In summary, one comes to the potential price function of the following format: $P_t = L(F_t, \Theta_t, \hat{F}_t^U, \hat{F}_t^M, \bar{\bar{F}}_t)$.

### 4.2.2. The Importance of $S_{U,t}$

From the information point of view, there are two types of signals. The private signal such as $S_{M,t}$ is only known by more informed agents I and M. The public signals including $P_t, D_t, S_{U,t}$ are known by all the agents. It seems a bit redundant and awkward to include $S_{U,t}$ as a public signal. Indeed as the benchmark asymmetric information case shows, it is not needed for the asymmetric information case when the agents I know the true underlying state. However this is not the case for higher order beliefs. For an equilibrium with higher order beliefs, the agents U have to have an additional signal. We have already discussed this in the two-period case in section 2. The following is the same reason applied in the current dynamic setting.

The information set of agents U includes history of public signals $P_t$ and $D_t$. When updating their beliefs about $F_t$ and $\Theta_t$ as well as $\hat{F}_t^M$, U agents need to use their information set through Kalman filter. If there is a history of only two public signals, the three beliefs of U agents are not linear independent to each other. As a result, $\bar{\bar{\bar{F}}}_t$ is not linear independent to $\hat{F}_t^U, \hat{\Theta}_t^U$. This reduction of higher order beliefs to first-order beliefs is different from the situation discussed in He and Wang (1995). In that situation, the reduction is from the fact that there is no more change in the underlying fundamental state variables after the initial time. Thus the higher order beliefs in the future dates can be reduced to the first-order beliefs about initial states. As Makarov and Rytchkov (2012) shows, this reduction breaks down if the states $\mathrm{F_t}$ and $\Theta_t$ are time varying. In our case, the reduction of higher order beliefs can be potentially from not enough linear independent information sources. Specifically, with only two linear independent signals, there can only be two linear independent beliefs. This requirement to have enough number of public signals to support the existence of higher order beliefs holds for general hierarchical information structure, which we will discuss in the later discussion section. Generally speaking, the highest order beliefs come from the least informed agents, who have to form beliefs not only on the fundamentals, but also on other more informed agents' beliefs. To support the existence of linear independent beliefs, the number of public signals should be large enough.

#### 4.2.3. Equilibrium

Summarizing the heuristics of price format and the requirement of enough public signals, the following is the first main result of the paper.

**Theorem 1**.

*There exists an equilibrium with a linear price function*:

$$P_t = p_F F_t - p_\Theta \Theta_t + p_M \hat{F}_t^M + p_{U1} \hat{F}_t^U + p_{U2} \bar{\bar{F}}_t \tag{35}$$

*where*

$$p_F = \frac{a_F}{R - a_F} - (p_M + p_{U1} + p_{U2}). \tag{36}$$

We can also represent the stock price as a function of the forecasting errors:

$$\begin{aligned} P_t &= \frac{a_F}{R - a_F} F_t - p_\Theta \Theta_t + p_M \delta_{F,t}^M + (p_{U1} + p_{U2}) \delta_{F,t}^U + p_{U2} \hat{\delta}_{F,t}^U \\ &= \frac{a_F}{R - a_F} F_t - p_\Theta \Theta_t + (p_M + p_{U2}) \delta_{F,t}^M + p_{U1} \delta_{F,t}^U + p_{U2} \Delta_{F,t}^U. \end{aligned} \tag{37}$$

The last expression of the price function represent the price function as a function of the true underlying states and the forecasting errors of agents. The forecasting errors include both agents' forecasting errors of underlying states, as well as the forecasting errors of U agents' beliefs about M agents' beliefs. As such the economy is indeed affected by the second-order beliefs.

In the following we heuristically derive the above results with the detailed proofs in the appendix. Assuming the existence of the linear equilibrium with the unknown coefficients, agents learn about the states from their respective information set. They then form optimal portfolio choices based on their beliefs about the dynamics of the states. Finally the market clearing condition gives the equilibrium stock prices, which is used to solve for the unknown constants in the price function.

We start with the beliefs of the agents. The informed agents I know about the fundamentals $F_t, \Theta_t$. Both agents U and M take the observables at each time and update their beliefs of the underlying states according to Kalman filter.

For agents M, the underlying states are $\xi_t^M \equiv (F_t, \Theta_t)$. The observables are $(P_t, D_t, S_{M,t}, S_{U,t})$, or equivalently $(\Pi_t^M, D_t, S_{M,t}, S_{U,t})$. Using the evolution of the states $\xi_t^M$, one can show that the evolution of the forecasting error of agents M follows:

$$\delta_{F,t+1}^M = a_\delta^M \delta_{F,t}^M + b_\delta^M \epsilon_{t+1}, \tag{38}$$

where $a_\delta^M$ is a constant, $b_\delta^M$ is a one by five constant vector, and $\epsilon_t$ is the five by one vector defined before.

For agents U, the underlying states are $(F_t, \Theta_t, \hat{F}_t^M)$. Note that the beliefs of agents M, $\hat{F}_t^M$, becomes a state variable for agents U. The observables are $(P_t, D_t, S_{U,t})$, or equivalently $(\Pi_t^U, D_t, S_{U,t})$. The agents U know about the dynamics of the first two state variables. From the above evolution of the forecasting errors of agents M, the evolution of $\hat{F}_t^M$ follows:

$$\hat{F}_{t+1}^M = (a_F - a_\delta^M) F_t + a_\delta^M \hat{F}_t^M + b_F^M \epsilon_{t+1} \tag{39}$$

where

$$b_F^M = b_\delta^M + (1, 0, 0, 0, 0). \tag{40}$$

Again using the Kalman filter, the evolution of the forecasting errors of agents U are:

$$\delta_{F,t+1}^U = a_\delta^U \delta_{F,t}^U + c_\delta^U \Delta_t^U + b_\delta^U \epsilon_{t+1} \tag{41}$$

$$\Delta_{F,t+1}^U = a_\Delta^U \delta_{F,t}^U + c_\Delta^U \Delta_t^U + b_\Delta^U \epsilon_{t+1}.$$

Having derived the evolution of the beliefs of the agents, we can calculate the expected share excess returns and their variance. Using the price formula, the share excess return is:

$$Q_{t+1} = (p_M + p_{U2})\left(a_\delta^M - R\right)\delta_{F,t}^M + \left[p_{U1}\left(a_\delta^U - R\right) + p_{U2}a_\Delta^U\right]\delta_{F,t}^U \tag{42}$$
$$+\left[p_{U1}c_\delta^U + p_{U2}(c_\Delta^U - R)\right]\Delta_t^U - p_\Theta(a_\Theta - R)\Theta_t + b_Q\epsilon_{t+1},$$

Where

$$b_Q = (p_F + 1, -p_\Theta, 1, 0, 0) + (p_M + p_{U2})b_\delta^M + p_{U1}b_\delta^U + p_{U2}b_\Delta^U. \tag{43}$$

Given the above excess returns, the agents' beliefs about the share excess returns are:

$$E_t^U(Q_{t+1}) = -p_\Theta(a_\Theta - R)\hat{\Theta}_t^U + (p_M + p_{U2})\left(a_\delta^M - R\right)\hat{\delta}_{F,t}^U$$
$$E_t^M(Q_{t+1}) = -p_\Theta(a_\Theta - R)\hat{\Theta}_t^M + \left[p_{U1}\left(a_\delta^U - R\right) + p_{U2}a_\Delta^U\right](\delta_{F,t}^U - \delta_{F,t}^M)$$
$$+ \left[p_{U1}c_\delta^U + p_{U2}(c_\Delta^U - R)\right]\Delta_t^U$$

The agents I know all the information in the economy including all other agents' beliefs, the expected share excess return of the agents I is simply the first four terms in equation (42) without the error terms. One can easily calculate the conditional variances of each agent as well.

Given the expected share excess returns and their variance, the optimal demand of the agents are of the similar form:

$$\theta^i = g_M^i\delta_F^M + g_{U1}^i\delta_F^U + g_{U2}^i\Delta^U + g_\Theta^i\Theta, \qquad \mathrm{i} = \mathrm{I}, \mathrm{U}, \mathrm{M}.$$

Finally using the market clearing condition $\omega_I\theta^I + \omega_U\theta^U + \omega_M\theta^M = \Theta$ , and let the coefficients of each term, $\delta_F^M, \delta_F^U, \Delta^U, \Theta$, to be equal, one can solve for the four unknown constants in the price function.

From the above derivation, we can see that even without the informed agents I, it is still possible to support an equilibrium with the price affected by the higher order beliefs of the agents U.

**Corollary 1**.

*Suppose only type U and M agents exist in the economy with* $\omega_U + \omega_M = 1, \omega_U, \omega_M > 0$. *There exists an equilibrium with a linear price function*:

$$P_t = p_F F_t - p_\Theta^{(2)} \Theta_t + p_M^{(2)} \hat{F}_t^M + p_{U1}^{(2)} \hat{F}_t^U + p_{U2}^{(2)} \bar{\bar{F}}_t$$

*where* $p_F$ *is defined in the Theorem 1.*

That a dynamic equilibrium with higher order beliefs exists even without fully informed agents is very interesting. In reality, the assumption with more-or-less informed agents seems to be more plausible than the informed-uninformed agents assumption used in the current asymmetric information literature. What we just showed is that as soon as the more informed agents form their beliefs, the second order beliefs from the less informed agents will affect the equilibrium prices and individual agents' holdings. Of course, we still need to have three public signals (the price, the dividend, and an additional signal) for the generic second order belief to exist.

## 5. Prices, Expected Share Returns, Individual Stock Demand and Trading Volumes

In this section, we discuss in details the rich results offered in our higher order belief model.

### 5.1. Equilibrium Stock Prices

The equilibrium stock is a function of five state variables, two fundamentals in $(F, \Theta)$, two first-order beliefs (on $F$) of the agents M and U, and one second-order belief (on $F$) of the agents U. Figure 1 shows the five coefficients as functions of the proportions of the informed and uninformed agents I, U in the population, $(\omega_I, \omega_U)$.

[Insert Figure 1 here]

Since the weights of the three types of agents sum up to one, the range of $(\omega_I, \omega_U)$ must be a triangle on the two dimensional surface. This is indicated in Panel A. The three corners represent the three extreme cases of only one type of agents in the economy (symmetric information cases): The one on the north is the case of all informed agents I; The one on the east is the case of all uninformed agents U; And the origin is the case of all partially informed agents M. In our setup, both agents U and M are partially informed. The standard literature on asymmetric information is either between agents I and M or between I and U under our setup. Thus the pattern for asymmetric information under two types of agents should be along either $\omega_U = 0$ or $\omega_M \equiv 1 - \omega_I - \omega_U = 0$.

Panel B to F show the five coefficients. Let's start with the symmetric information cases. Proposition 1 states that when all the agents are informed, the equilibrium price is a function of the fundamentals $(F, \Theta)$ only. This is clearly seen from the five panels: For the point $\omega_I = 1$, the only nonzero (positive) coefficients are $p_F, p_\Theta$. Proposition 2 states that when agents are partially informed, either $\omega_U = 1$ or $\omega_M$=1, the equilibrium price is a function of the agents' beliefs about $F$ , $\hat{F}$, and the actual share supply, $\Theta$. Indeed the only non-zero coefficients are from Panel C (all M agents) or Panel D (all U agents), and Panel F ($\Theta$). Of course, the second-order beliefs do not play a role here under the symmetric information.

For the asymmetric information case that has been studied extensively in the literature, the proposition 3 states that the equilibrium price is a function of the two fundamentals plus one first-order belief from the partially informed agent. This is clear from the panels for either either $\omega_U = 0$ or $\omega_M = 0$.

Interestingly, if there is no informed agents, $\omega_I = 0$, the corollary 1 states that the equilibrium stock price is a function of five state variables. In other words, simply requiring hierarchical information structure is not enough to prevent second-order beliefs. As long as no agents in the economy know about the fundamentals and thus have to form beliefs on them, the less informed agents potentially have to form beliefs about beliefs. This is specifically striking from Panel B. Recall in the symmetric partial informed agents case (M or U), the equilibrium price is not a function of fundamentals because the first order beliefs of the fundamentals is a sufficient statistics to characterize the economy. But if there are two types of partially informed agents with hierarchical information structure, two first-order beliefs and one second-order beliefs do not generally form a set of sufficient statistics to characterize the economy. In other words, simply taking more informed agents' beliefs as the "fundamentals" is too restrictive.[11]

The messages about the effect of first-order beliefs on the prices are quite intuitive. Panel B shows that when $F_t$ is high, the price is higher than that in the symmetric partially informed case (zero effect under all M or U), but lower than that in the symmetric full informed case (all I). Panel C and D show that when there are more agents of one (partially informed) type, the effect of that group's agents' beliefs are stronger. Panel F shows that when there is more share supply, the effect of the share supply on the price is lower than that under partial but symmetric information case. The reason is that under symmetric partial information case, the effect of (higher) share supply is purely a (negative) liquidity effect, resulting in lower stock price. Under asymmetric information case, it has additional

[11] A comparison between these two equilibria is an interesting further study.

information effect, which mitigate the liquidity effect. Lower stock price may be from lower fundamentals instead of higher stock supply.

The novel part of our model is from Panel E. This shows the effect of second-order beliefs of the uninformed agents on the stock prices. As one can see, the effect is negative. In other words, when uninformed agents U believe that partially informed agents M has high expectation about fundamentals, the price is dampened. The intuition is that when this happens, it is also the case that agents U likely over-estimate agents M's belief about the fundamental *F*. Thus U will likely over-sell (or under-buy) the shares to dampen the stock prices.

Even though we don't do calibration exercises in this paper, one cannot help wondering the small number of the coefficient, $p_{U2}$. As we will see, even though the price coefficient of second-order belief is small, the effect of the second-order belief on the stock demand and trading is much larger because the variance is small. Thus the effects on the stock demand and trading are nontrivial.

### 5.2 Expected Share Excess Return and Individual Stock Demand

Since we use CARA-normal setup, we will mainly discuss the expected share excess returns instead of expected dollar excess returns. In our setting, the effect of *F* on the share excess returns is always zero. There are no predictability from current $F_t$ to next period returns since it is already priced. The shocks on the share excess returns are mainly from the forecasting errors on *F*, both first-order and second-order. The effect of $\Theta$ is mainly from its liquidity effect. Figure 2 presents the share excess returns for the informed investors I.

[insert Figure 2 here]

The top two panels in Figure 2 present the effects of estimation errors of both partially informed agents U and M on the expected returns. One can see that they are both negative, meaning that when agents I observe that either U or M overestimate $F$, they expect a lower return in the future. The intuition is that either U or M over-allocate in the stock, thus pushing up the current price.

The left bottom panel shows the effect of the estimation error of agents U's second-order beliefs on agents I's expected excess returns. It is positive. So if U over-estimates M's expectation of $F$, the expected excess returns are higher. The intuition is that U will over-sell stocks because U thinks that M over-estimates $F$ and thus over-buy the stock.

Recall in the CARA-normal setting, the stock demand is proportional to the expected share excess returns. So the above pattern is similar as the demand of agents I, with the exception of a constant (the product of risk aversion and the variance of the share excess returns). Figure 3 and 4 thus report the individual stock demand by agents U and M.

[insert Figure 3 here]

[insert Figure 4 here]

Note that for agents U and M, their state variables are different from those of agents I because they have different information set from each other. For example, the state variables of agents M include two of their own first-order beliefs about the fundamentals $(F, \Theta)$ and agents U's first and second-order beliefs about $F$. The state variables of agents U include two of their own first-order beliefs about the fundamentals $(F, \Theta)$ and U's belief about the forecasting error of M. Nevertheless, we want to plot the individual demand of M and U from agents I's point of view, because this way we can see clearly the effect of underlying state variables (c.f. their perceived state variables) on agents M and U's expected share excess returns and thus their demand.

From Figure 3 top left panel, we can see that effect of M's forecasting error on M's demand is drastically different from those of I and U. Here the effect of M's forecasting error on M's demand goes to zero when all the agents are of M type. This is just like the expected share excess returns that has zero coefficient of *F*. The effect has already been incorporated into the prices. From M's point of view, the stock price already incorporate agents M's beliefs. Thus the expected share excess returns has zero coefficients on agents M's own beliefs (thus forecasting errors). Otherwise agents I and M have similar pattern over the two forecasting errors of agents U: Agents I and M both know about the forecasting errors of agents U.

Figure 4 reports the results for agents U. The top two panels are similar as those for agents I, because agents U's expectation of agents M's forecasting error is the sum of agents' forecasting error plus U's forecasting errors on M's forecasting (plus one more term). Thus the coefficient from this term are the same. However, the left bottom panel indicates the drastic different effect from the agents U's forecasting errors on agents M's beliefs about *F*. As stated before, when agents U overestimate M's beliefs, agents U think that M over-buy the stock, thus agents U want to over-sell the stock (thus the negative coefficient). Agents I and M, however, know that agents U make mistake in over-estimating and thus over-sell. So they want to buy in return.

Finally, even though the pattern on Θ seems to be quite different from each other, but we tend to think that they are essentially the same by checking their actual numbers.

### 5.3 Group Demand and Trading Volume

Even though we discuss the expected share excess return and individual demand of stocks for three types of investors, the actual trading depends on the overall demand from each group of investors. In our current setting in which each generation only lasts one period (two dates), the group holdings correspond to the trading from old generation to the new generation. This is of course not ideal. In the next section we briefly discuss the extension to the long-lived agents case which resolve this problem. Figures 5a-d report the group demand coefficients of the five state variables. As stated before, we discuss all the agents' demand on the five state variables of the agents I. This is more so in the current discussion because we should see that all the group demand should sum up to zero with exception of the state variable Θ, which should be equal to one.

Figure 5a reports the effect of M's forecasting error of *F* on the group demand. One can see that when there is a positive shock to agents M's forecasting error, agents M will be a net buyer while agents I and U will be the net sellers, *ceteris paribus*. Under the current asymmetric information literature, the study on the trading volume is essentially a study on the changing of the stock holdings of one type of investors, because there are only two types of investors in the market. In our setting, this is much richer. Even though the trading of one group of investors is equal to the sum of the signed trading of the other two groups, the set of long/short groups can change.

[insert Figure 5a here]

Figure 5b reports the effect of U's forecasting error of *F* on the group demand. Similarly to the above case, when U over estimate, U will over-allocate while I and M will under-allocate, *ceteris paribus*. However, the long side changes from agents M to agent U.

[insert Figure 5b here]

Figure 5c reports the effect of U's second-order forecasting error on the group demand. This is the most interesting and yet most complicated one. Using the same intuition as before, agents U are net seller and agents I and M are net buyers, *ceteris paribus*. However, the main trading here occurs between agents U and M, the effect on agents I is almost nil.

[insert Figure 5c here]

Finally figure 5d reports the effect of share supply on the group demand. This is quite trivial in our setting since it looks like to be linear and roughly allocate among the three groups. This implies that the nonlinear effect from the previous discussion about different beliefs are not likely caused by the shocks to the aggregate share supply.

[insert Figure 5d here]

## 6. Further Discussion

There are different directions that this model can be easily generalized.

### 6.1 Intertemporal Consumption with Long-Lived Agents.

In the main text we use OLG model to simplify the discussion. The advantage of OLG model is that the stock demand is proportional to the expected share excess returns. Thus the relationship between holding and returns are quite trivial. However, there are two serious problem with this assumption. One is the assumption that new generation inheret the information from the old generation. The other is the difference between holdings and trading volume. This can be resolved if we use long-lived agents model. Here we briefly discuss this extension and argue that methodologically there are no difference between long-lived agent model and OLG model.

In the long-lived agent model, we still have three groups of agents. However, instead of maximizing a one-period model, now agents have consumption over infinite horizon. We still assume time separable CARA utility function for each agent and they are the same across the agents with identical subjective time discount factor and risk aversion coefficient. Thus all the differences among the agents come from their information structure.

We still keep the same hierarchical information structure among agents I, U and M. Now the inheritance of information is not an issue. The learning and belief updating process does not change compared with the OLG setup. The complication mainly come from the agents' utility maximization problem. However, given CARA-normal assumption, this is tedious but not challenge.

Specifically, we assume, as in Wang (1993, 1994), that each agents' value function is negative exponential in their wealth and a quadratic form of their other respective state variables. Then we can solve the Bellman equation analytically. The optimal share holdings of each agent are still a linear function of the agent's state variable, even though the coefficients are much more complex. Market clearing conditions solve the unknown coefficients in the price functional.

### 6.2 Beliefs of Order Higher than Two

It is straight forward to generalize the model to incorporate beliefs of order higher than two. For example, we can introduce three partially informed agents, A, B, C and still keep the hierarchical information structure[12]. Specifically, let agents C receive some signals $S_{C1}, S_{C2}, S_{C3}$, agent B receives all the $S_{Ck}, k = 1,2,3$ and $S_B$, and agents A receive all the $S_{Ck}, k = 1,2,3$, $S_B$ and $S_A$. Here we ignore the time subscripts if there is no confusion.

The information structure is hierarchical. The information set of A includes that of B, which includes that of C. The equilibrium price is a function of fundamental state variables $\{F, \Theta\}$ and derived states of the beliefs of the agents. Specifically, it generally includes first-order beliefs, $\{\hat{F}^A, \hat{\Theta}^A\}$ of A, and the first-order beliefs, $\{\hat{F}^B, \hat{\Theta}^B\}$ and second-order beliefs $\{\bar{\bar{F}}^{AB}, \bar{\bar{\Theta}}^{AB}\}$ of agents B. This is identical to the situation discussed in the main text, in which the second-order beliefs of B are the B's beliefs of the A's beliefs of $\{F, \Theta\}$.

[12] Just like two partially informed agents with hierarchical information structure support an equilibrium with second-order beliefs (corollary 1), here with three partially informed agents the economy supports an equilibrium with third-order beliefs.

It also includes the first-order beliefs the first-order beliefs $\hat{F}^C, \widehat{\Theta}^C$ and second-order beliefs $\bar{\bar{F}}^{AC}, \bar{\bar{\Theta}}^{AC}, \bar{\bar{F}}^{BC}, \bar{\bar{\Theta}}^{BC}$, and third order beliefs $\tilde{F}^{ABC}, \tilde{\Theta}^{ABC}$. In this case, there are four second-order beliefs of C about the first-order beliefs of both A and B. There are two third-order beliefs of C about the second-order beliefs of B.

Thus in general the price function is a linear function of 16 state variables: Two fundamentals $F, \Theta$, two first-order beliefs of A, four beliefs (two first-order, two second-order) of B, eight beliefs (two first-order, four second-order and two third-order) of C. However, not all the fourteen beliefs are linear independent. We will show, like in the main text, the set of the non-redundant state variables includes two fundamentals, and all the beliefs on F only.

Start with agents A. Observing the prices P , agents A knows about $\Pi^A \equiv p_F F - p_\Theta \Theta$. Since agents A knows about all the other beliefs but not the true underlying states. Taking expectation of $\Pi^A$ with respect to agent A's information set. It follows that

$$p_F F - p_\Theta \Theta = \Pi^A = E^A(\Pi^A) = p_F \hat{F}^A - p_\Theta \hat{\Theta}^A.$$

Thus $\widehat{\Theta}^A = \Theta + \frac{p_F}{p_\Theta}(\hat{F}^A - F)$. Namely $\hat{\Theta}^A$ is redundant.

Furthermore, agents B and C know about this relationship, thus by taking expectation with respect to the respective information set of agents B and C:

$$\bar{\bar{\Theta}}^{AB} = \hat{\Theta}^B + \frac{p_F}{p_\Theta}\left(\bar{\bar{F}}^{AB} - \hat{F}^B\right)$$

$$\bar{\bar{\Theta}}^{AC} = \hat{\Theta}^C + \frac{p_F}{p_\Theta}\left(\bar{\bar{F}}^{AC} - \hat{F}^C\right).$$

So $\bar{\bar{\Theta}}^{AB}$ and $\bar{\bar{\Theta}}^{AC}$ are redundant as well.

Even more, agents C also know about the above second-order beliefs $\bar{\bar{\Theta}}^{AB}$ of agents B. By taking expectation with respect to agents C's information set, it follows the third-order beliefs of agents C, $\tilde{\Theta}^{BC}$ follows: $\tilde{\Theta}^{ABC} = \bar{\bar{\Theta}}^{BC} + \frac{p_F}{p_\Theta}\left(\tilde{F}^{ABC} - \bar{\bar{\Theta}}^{BC}\right)$. Thus $\tilde{\Theta}^{ABC}$ is redundant. Combining the results so far, the pricing function P reduces to a linear function of $F$, $\Theta$, all the beliefs of F, and $\hat{\Theta}^B, \bar{\bar{\Theta}}^{BC}, \hat{\Theta}^C$.

Now let us check agents B. By observing the price function, agents B knows about the following: $\Pi^B \equiv p_F F - p_\Theta \Theta + p_{\hat{F}^A}\hat{F}^A$. Since agents B know about all the beliefs of agents C. Again by taking the expectation of $\Pi^B$ with respect to B's information set, it follows that:

$$\hat{\Theta}^B = \Theta + \frac{p_F}{p_\Theta}\left(\hat{F}^B - F\right) + \frac{p_{\hat{F}^A}}{p_\Theta}\left(\bar{\bar{F}}^{AB} - \hat{F}^A\right).$$

Thus $\hat{\Theta}^B$ is redundant. The agents C know about this relationship. By taking expectation with respect to C's information set, it follows:

$$\bar{\bar{\Theta}}^{BC} = \hat{\Theta}^C + \frac{p_F}{p_\Theta}\left(\bar{\bar{F}}^{BC} - \hat{F}^C\right) + \frac{p_{\hat{F}^A}}{p_\Theta}\left(\tilde{F}^{ABC} - \bar{\bar{F}}^{AC}\right).$$

Thus $\bar{\bar{\Theta}}^{BC}$ is redundant.

Finally for agents C, observing the price P is equivalent to observing the following:

$$\Pi^C \equiv p_F F - p_\Theta \Theta + p_{\hat{F}^A}\hat{F}^A + p_{\hat{F}^B}\hat{F}^B + p_{\bar{\bar{F}}^{AB}}\bar{\bar{F}}^{AB}.$$

Taking expectation with respect to agents C's information set, it follows that $\hat{\Theta}^C$ is also redundant.

Summarize all the results together, the equilibrium price is a linear function of $\{F, \Theta, \hat{F}^A, \hat{F}^B, \hat{F}^C, \bar{\bar{F}}^{AB}, \bar{\bar{F}}^{AC}, \bar{\bar{F}}^{BC}, \tilde{F}^{ABC}\}$.

There are two implications from the above discussions. First, the higher order beliefs are generic. As discussed in the main text, in the case of only two partially informed types agents say A and B, the second order belief $\bar{\bar{F}}^{AB}$ is a linear combination of $F, \Theta, \hat{F}^A, \hat{F}^B, \hat{\Theta}^B$.

Even though we use $\bar{\bar{F}}$ as a state variable, one can equivalently use $\widehat{\Theta}^B$ instead. Thus one may argue that the equilibrium price in the main text is still a function of the first-order beliefs. The above discussion indicates that the higher order beliefs are generic. If one uses second-order beliefs of $\Theta$ to substitute the third-order beliefs of , this cannot be further reduced. Instead, keeping only beliefs on one fundamental state ( $F$ in this paper) makes the results much cleaner.

Second, for agents C and B to have higher order beliefs, they have to receive linear independent information sources other than the two public signals, $P$ and $D$ . For agents B, an additional signal $S_B$ is needed because agents B have to learn about three underlying state variables, $F, \Theta, \hat{F}^A$. For agents C, three additional signals, $S_{Ck}, k = 1, 2, 3,$ are needed because agents C have to learn about five underlying state variables, $F, \Theta, \hat{F}^A, \hat{F}^B, \bar{\bar{F}}^{AB}$.

## 6.2 The Information Content of Public Signals

We have shown that to support higher order beliefs, less informed agents should receive additional signals besides $P$ and $D$. So far we have assumed that the signals are just a linear function of fundamental F plus noise. However, this does not have to be the case. The purpose of the signals is to be used by the agents to update their beliefs, as long as they are linear independent. Thus the signals received by the agents can be a linear function of underlying states plus noises. This means that the signals received by the agents can be signals about the beliefs of other (more informed) agents. It does not change the main results of the paper on the format of the equilibrium price function. However, it will change the actual coefficients and the updating process of the beliefs of agents.

## 7. Conclusion

We study a dynamic noisy rational expectation model with higher order beliefs. The generalization from two agent cases to three agent cases is not as trivial. Realistically speaking partially informed assumption seems to be more plausible. The potential gain from the richness of the model seems to be enormous. The current baseline model benefits enormously from the simplified assumption on the agents' maximization problem. To carry it to the data requires even more realistic setup. Further work is needed to move along this direction.

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## Appendix: Proofs

**Proposition 1**.

Assume that the price function has the following format:

$P_t = -p_0\Theta_t + p_1F_t.$

It follows that

$$\begin{aligned} Q_{t+1} &= P_{t+1} + D_{t+1} - RP_t \\ &= [(1+p_1)a_F - Rp_1]F_t + (R - a_\Theta)p_0\Theta_t + (1+p_1)\epsilon_{F,t+1} + \epsilon_{D,t+1} - p_0\epsilon_{\Theta,t+1}. \end{aligned}$$

Thus

$$\begin{aligned} E_t(Q_{t+1}) &= (R - a_\Theta)p_0\Theta_t + [(1+p_1)a_F - Rp_1]F_t \\ \sigma^2(Q_{t+1}) &= (1+p_1)^2\sigma_F^2 + \sigma_D^2 + p_0^2\sigma_\Theta^2. \end{aligned}$$

Substituting this into the demand function and aggregate over agents:

$$\Theta_t = \frac{(R - a_\Theta)p_0\Theta_t + [(1+p_1)a_F - Rp_1]F_t}{\gamma[(1+p_1)^2\sigma_F^2 + \sigma_D^2 + p_0^2\sigma_\Theta^2]}.$$

For this to hold under any $F_t$ and $\Theta_t$, the resulting coefficients must hold in the proposition.

Q.E.D.

### Kalman Filter with Correlated Errors

To derive the updated beliefs for the following results, we need to consider generalization of Kalman filter to the situation that the error terms in the state and observation equations are correlated. The straight forward calculation leads to the following:

*Proposition (Kalman Filter with Correlated Errors) Assume that the state vector $\xi_t$ and observable vector $y_t$ follow:*

$$\begin{aligned} \xi_{t+1} &= M\xi_t + v_{t+1} \\ y_t &= H'\xi_t + w_t, \end{aligned}$$

where $E(v_tv_t') = Q, E(w_tw_t') = A, E(v_tw_t') = B$, and all other covariance matrices are zero. Note here $y_t, \xi_t$ are the vectors not necessarily with the same lengths. Then the evolution of the expected value of $\xi_t$ are the following:

1. Start with the current expected value of state, $\hat{\xi}_{t|t} \equiv E_t(\xi_t)$, and the covariance matrix, $P_{t|t} \equiv E_t[(\xi_t - \hat{\xi}_{t|t})(\xi_t - \hat{\xi}_{t|t})']$.
2. The predicted future states are:

$$\begin{aligned} &\hat{\xi}_{t+1|t} = M\hat{\xi}_{t|t}, \\ &\xi_{t+1} - \hat{\xi}_{t+1|t} = M(\xi_t - \hat{\xi}_{t|t}) + v_{t+1} \\ &P_{t+1|t} = E\left[(\xi_{t+1} - \hat{\xi}_{t+1|t})(\xi_{t+1} - \hat{\xi}_{t+1|t})'\right] = MP_{t|t}M' + Q. \end{aligned}$$

3. The expectation of $y_{t+1}$ are:

$$\hat{y}_{t+1|t} = H'\hat{\xi}_{t+1|t}$$
$$y_{t+1} - \hat{y}_{t+1|t} = H'(\xi_{t+1} - \hat{\xi}_{t+1|t}) + w_{t+1}$$
$$E\left[(\xi_{t+1} - \hat{\xi}_{t+1|t})(y_{t+1} - \hat{y}_{t+1|t})'\right] = P_{t+1|t}H + B$$
$$E[(y_{t+1} - \hat{y}_{t+1|t})(y_{t+1} - \hat{y}_{t+1|t})'] = H'P_{t+1|t}H + R + H'B + B'H.$$

4. After observing $y_{t+1}$, the updated expectation of the states $\xi_{t+1}$ is then:

$$\hat{\xi}_{t+1|t+1} = \hat{\xi}_{t+1|t} + (P_{t+1|t}H + B)(H'P_{t+1|t}H + R + H'B + B'H)^{-1}(y_{t+1} - \hat{y}_{t+1|t})$$
$$P_{t+1|t+1} = P_{t+1|t} - (P_{t+1|t}H + B)(H'P_{t+1|t}H + R + H'B + B'H)^{-1}(H'P_{t+1|t} + B').$$

**Proposition 2**.

First let's study the updating process of beliefs after observing $D_t$. From the above Kalman filter process, one can use the following mapping: $\xi_t = F_t, y_t = (D_t, S_{U,t})'$. It follows that $M = a_F$, $H = (1, 1)$, $Q = \sigma_F^2$, $A = diag(\sigma_D^2, \sigma_B^2)$ and $B = 0$. The stationary $\Omega \equiv P_{t+1|t+1} = P_{t|t}$ is a constant. The updating process of beliefs, $\hat{F}_{t|t}$, again follows from the Kalman filter. The relevant updating variables are (taking into account the stationary property):

$$P_{t+1|t} = a_F^2\Omega + \sigma_F^2$$
$$E[(\xi_{t+1} - \hat{\xi}_{t+1|t})(y_{t+1} - \hat{y}_{t+1|t})'] = (a_F^2\Omega + \sigma_F^2)(1, 1)$$

$$E\left[(y_{t+1} - \hat{y}_{t+1|t})(y_{t+1} - \hat{y}_{t+1|t})'\right] = \begin{pmatrix} a_F^2\Omega + \sigma_F^2 + \sigma_D^2 & a_F^2\Omega + \sigma_F^2 \\ a_F^2\Omega + \sigma_F^2 & a_F^2\Omega + \sigma_F^2 + \sigma_U^2 \end{pmatrix},$$

where (1, 1) in the second equation indicates the (1, 1) component in the two by two matrix.

Thus the stationary variance $\Omega$ satisfies:

$$(\Gamma_D + \Gamma_U)a_F^2\Omega^2 + [(\Gamma_D + \Gamma_U)\sigma_F^2 + 1 - a_F^2]\Omega - \sigma_F^2 = 0,$$

where $\Gamma_D \equiv \frac{1}{\sigma_D^2}, \Gamma_U \equiv \frac{1}{\sigma_U^2}$ are the precision of $D_t$ and $S_{U,t}$ respectively.

The updating of the belief about the $F_t$ follows:

$$\hat{F}_{t+1|t+1} = a_F\hat{F}_{t|t} + \frac{1}{\Gamma_D + \Gamma_B + \Gamma_F}\left[\Gamma_D(D_{t+1} - a_F\hat{F}_{t|t}) + \Gamma_U(S_{U,t+1} - a_F\hat{F}_{t|t})\right],$$

where $\Gamma_F \equiv \frac{1}{a_F^2\Omega+\sigma_F^2}$.

Assuming the equilibrium price to be of the following form:

$$P_t = -p_0\Theta_t + p_1\hat{F}_{t|t}.$$

Given this expression, it follows that the excess share demand,

$$Q_{t+1} = P_{t+1} + D_{t+1} - RP_t$$

$$= p_0(R - a_\Theta)\Theta_t + [a_F - (R - a_F)p_1]\hat{F}_{t|t} + \frac{a_F[(\Gamma_D + \Gamma_U)(1 + p_1) + \Gamma_F]}{\Gamma_D + \Gamma_U + \Gamma_F}\left(F_t - \hat{F}_{t|t}\right) + \frac{(\Gamma_D + \Gamma_U)(1 + p_1) + \Gamma_F}{\Gamma_D + \Gamma_U + \Gamma_F}\epsilon_{F,t+1} - p_0\epsilon_{\Theta,t+1} + \frac{(1 + p_1)\Gamma_D + \Gamma_U + \Gamma_F}{\Gamma_D + \Gamma_U + \Gamma_F}\epsilon_{D,t+1} + \frac{p_1\Gamma_U}{\Gamma_D + \Gamma_U + \Gamma_F}\epsilon_{U,t+1}.$$

With this result, it follows that:

$$E_t(Q_{t+1}) = p_0(R - a_F)\Theta_t + [a_F - (R - a_F)p_1]\hat{F}_{t|t}$$

$$\sigma^2(Q_{t+1}) = \frac{1}{(\Gamma_D + \Gamma_U + \Gamma_F)^2}\Big[a_F^2\big((\Gamma_D + \Gamma_U)(1 + p_1) + \Gamma_F\big)^2\Omega + \big((\Gamma_D + \Gamma_U)(1 + p_1) + \Gamma_F\big)^2\sigma_F^2 + (\Gamma_D + \Gamma_U + \Gamma_F)^2 p_0^2\sigma_\Theta^2 + \big((1 + p_1)\Gamma_D + \Gamma_U + \Gamma_F\big)^2\sigma_D^2 + p_1^2\Gamma_U\Big].$$

where in the last step we use $\Gamma_B\sigma_B^2 = 1$.

Substitute this into the demand function, aggregate and let the coefficients to match. The resulting price function will follow.

Q.E.D.

**Proposition 3.**

Since the informed agents I know about the true states of the economy, the evolution process of their beliefs are given by the state evolution process. The uninformed agents U only know about the prices, dividends and signal $S_U$. From these they want to infer the process of underlying state $\{F, \Theta\}$. As discussed in the main text, knowing $P_t$, uninformed agents knows about $\Pi_t \equiv p_1 F_t - p_\Theta \Theta_t$. It follows that there exists the following equivalence: $p_1 F_t - p_\Theta\Theta_t = p_1\hat{F}_t - p_\Theta\hat{\Theta}_t$. Or equivalently: $p_1\delta_{F,t} = p_\Theta\delta_{\Theta,t}$, and $\hat{\Theta}_t = \Theta_t + \frac{p_1}{p_\Theta}\delta_{F,t}$.

To use Kalman filter, let $\xi_t = (F_t, \Theta_t)'$, $y_t = (D_t, S_{U,t}, \Pi_t)'$, $v_t = (\epsilon_{F,t}, \epsilon_{\Theta,t})'$, $w_t = (\epsilon_{D,t}, \epsilon_{U,t}, 0)'$. It follows that $B = 0$ and:

$$M = \begin{pmatrix} a_F & 0 \\ 0 & a_\Theta \end{pmatrix}, H' = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ p_1 & -p_\Theta \end{pmatrix}, Q = \begin{pmatrix} \sigma_F^2 & 0 \\ 0 & \sigma_\Theta^2 \end{pmatrix}, A = \begin{pmatrix} \sigma_D^2 & 0 & 0 \\ 0 & 0 & \sigma_U^2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Define $K = (M\Omega M' + Q)H(H'(M\Omega M' + Q)H + A)^{-1}$, where $\Omega$ satisfies:

$$\Omega = (M\Omega M' + Q) - KH'(M\Omega M' + Q).$$

The updating of beliefs about state variables are:

$$\hat{\xi}_{t+1|t+1} = M\hat{\xi}_{t|t} + K(y_{t+1} - H'M\hat{\xi}_{t|t})$$
$$= M\hat{\xi}_{t|t} + KH'M(\xi_t - \hat{\xi}_{t|t}) + K(w_{t+1} + H'v_{t+1}).$$

This implies that (using $\xi_{t+1} = M\xi_t + v_{t+1}$):

$$\delta_{t+1} = (M - KH'M)\delta_t + Kw_{t+1} - (1 - KH')v_{t+1},$$

where $\delta_t \equiv (\delta_{F,t}, \delta_{\Theta,t})' \equiv \hat{\xi}_{t|t} - \xi_t$.

Using the result that $\delta_{\Theta,t} = \frac{p_1}{p_\Theta}\delta_{F,t}$, it follows that there exist the following relationship:

$$\delta_{F,t+1} = a_\delta \delta_{F,t} + b_\delta \epsilon_{t+1},$$

where $\epsilon_t = (\epsilon_{D,t}, \epsilon_{F,t}, \epsilon_{\Theta,t}, \epsilon_{U,t})$.

As such, the share excess return is then:

$$Q_{t+1} = P_{t+1} + D_{t+1} - RP_t$$
$$= [\alpha_F - (R - \alpha_F)(p_1 + p_2)]F_t - p_2(R - a_\delta)\delta_{F,t} + p_\Theta(R - a_\Theta)\Theta_t + b_Q\epsilon_{t+1},$$

where $b_Q\epsilon_{t+1} = (p_1 + p_2 + 1)\epsilon_{F,t+1} + p_2 b_\delta \epsilon_{t+1} - p_\Theta \epsilon_{\Theta,t+1} + \epsilon_{D,t+1}$.

For informed investors I:

$$E_t^I(Q_{t+1}) = [\alpha_F - (R - \alpha_F)(p_1 + p_2)]F_t - p_2(R - a_\delta)\delta_{F,t} + p_\Theta(R - a_\Theta)\Theta_t.$$

For uninformed investors U:

$$E_t^U(Q_{t+1}) = [\alpha_F - (R - \alpha_F)(p_1 + p_2)]\hat{F}_t + p_\Theta(R - a_\Theta)\hat{\Theta}_t.$$

Using the fact that $\hat{\Theta}_t = \Theta_t - \frac{p_1}{p_\Theta}\delta_{F,t}$ and the demand function of both agents, one can see that the demands of the two types of agents are:

$$\theta^I = f_1^I F_t + f_2^I \delta_{F,t} + f_3^I \Theta_t$$
$$\theta^U = f_1^U F_t + f_2^U \delta_{F,t} + f_3^U \Theta_t.$$

Aggregate over the agents: $\omega\theta^I + (1 - \omega)\theta^U = \Theta$ and let the coefficients of $F_t, \delta_{F,t}, \Theta_t$ to match, one can solve for $p_1, p_2, p_\Theta$. Thus the results.

Q.E.D.

**Theorem 1 and Corollary 1.**

We follow the steps of proof spelled out in the main text. First we obtain the beliefs of the agents about the dynamics of the underlying state variables, specifically the beliefs about the mean payoffs of the stock dividends. Next by using these beliefs, agents obtain the expected shares excess returns and the variances. These give us the demands of the agents. Finally by using the market clearing condition, we solve for the coefficients in the price function.

The informed agents I know every information in the economy. So their beliefs are just the underlying state processes of $(F, \Theta)$. About the two partially informed agents U and M, we first consider the more informed agents M, because their beliefs will become a state variable for the less informed agents U. The state variables are $\xi^M = (F, \Theta)'$; The observables are $y^M = (D, S_M, S_U, \Pi^M)$, where $\Pi^M = p_F F - p_\Theta \Theta$ . Using the notation from the Kalman filter before, define $v^M = (\epsilon_F, \epsilon_\Theta)'$; and $w^M = (\epsilon_D, \epsilon_M, \epsilon_U, 0)'$. It follows that $B = 0$ and

$$M^M = \begin{pmatrix} a_F & 0 \\ 0 & a_\Theta \end{pmatrix}, H^M = \begin{pmatrix} 1 & 1 & 1 & p_F \\ 0 & 0 & 0 & -p_\Theta \end{pmatrix}, Q^M = \begin{pmatrix} \sigma_F^2 & 0 \\ 0 & \sigma_\Theta^2 \end{pmatrix},$$

$$A^M = \begin{pmatrix} \sigma_D^2 & 0 & 0 & 0 \\ 0 & \sigma_M^2 & 0 & 0 \\ 0 & 0 & \sigma_U^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Define

$$K^M = (M^M \Omega^M (M^M)' + Q^M) H^M ((H^M)'(M^M \Omega^M (M^M)' + Q^M) H^M + A^M)^{-1},$$

where $\Omega^M$ satisfies $\Omega^M = (M^M \Omega^M (M^M)' + Q^M) - K^M (H^M)'(M^M \Omega^M (M^M)' + Q^M)$.

The updating of beliefs about state variables are:

$$\begin{aligned} \hat{\xi}^M_{t+1|t+1} &= M^M \hat{\xi}^M_{t|t} + K^M \left( y^M_{t+1} - (H^M)' M^M \hat{\xi}^M_{t|t} \right) \\ &= M^M \hat{\xi}^M_{t|t} + K^M (H^M)' M^M (\xi^M_t - \hat{\xi}^M_{t|t}) + K^M (w^M_{t+1} + (H^M)' v^M_{t+1}). \end{aligned}$$

From the above expression one can derive two expressions about the evolution of $\hat{F}^M$, the agent M's expectation about the underlying state F. First, one can represent it as the evolution of forecasting errors. Using $\xi^M_{t+1} = M^M \xi^M_t + v^M_{t+1}$ :

$$\delta^M_{t+1} = (M^M - K^M (H^M)' M^M) \delta^M_t + K^M w^M_{t+1} - (1 - K^M (H^M)') v^M_{t+1},$$

where

$$\delta^M_t = \left( \delta^M_{F,t}, \delta^M_{\Theta,t} \right)' \equiv \hat{\xi}^M_{t|t} - \xi^M_t.$$

Using the result that $\delta^M_{\Theta,t} = (p_F)/(p_\Theta) \delta^M_{F,t}$ , it follows that there exists the following relationship:

$$\delta^M_{F,t+1} = a^M_\delta \delta^M_{F,t} + b^M_\delta \epsilon_{t+1},$$

where $\epsilon_t = (\epsilon_{F,t}, \epsilon_{\Theta,t}, \epsilon_{D,t}, \epsilon_{M,t}, \epsilon_{U,t})'$, and $b^M_\delta = (b^M_{\delta 1}, b^M_{\delta 2}, b^M_{\delta 3}, b^M_{\delta 4}, b^M_{\delta 5})$ is a constant vector. This expression will be used in the market clearing condition.

Second, one can represent $\hat{F}^M_{t+1}$ as a linear function of $(F_t, \hat{F}^M_t)$ by using the evolution equations of $F_t$ in the above equations:

$$\hat{F}^M_{t+1} = \left(a_F - a^M_\delta\right) F_t + a^M_\delta \hat{F}^M_t + b^M_F \epsilon_{t+1},$$

where $b^M_F = b^M_\delta + (1, 0,0,0, 0)$ is a constant vector. This will be used in the following uninformed agent analysis since then $\hat{F}^M$ becomes a state variable itself.

For uninformed agents, the state variables are $(F, \Theta, \hat{F}^M)'$. Namely U agents need to form beliefs about both the underlying fundamentals and M agent's beliefs. The observables are $y^U = (D, S_U, \Pi^U)$, where $\Pi^U = p_F F - p_\Theta \Theta + p_M \hat{F}^M$. It follows that $v^U = (\epsilon_F, \epsilon_\Theta, b^M_F \epsilon)'$, $w^U = (\epsilon_D, \epsilon_U, 0)'$. Define the following:

$$M^U = \begin{pmatrix} a_F & 0 & 0 \\ 0 & a_\Theta & 0 \\ a_F - a^M_\delta & 0 & a^M_\delta \end{pmatrix}, \qquad (H^U)' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ p_F & -p_\Theta & p_M \end{pmatrix},$$
$$Q^U = \begin{pmatrix} \sigma_F^2 & 0 & b^M_{F1}\sigma_F^2 \\ 0 & \sigma_\Theta^2 & b^M_{F2}\sigma_\Theta^2 \\ b^M_{F1}\sigma_F^2 & b^M_{F2}\sigma_\Theta^2 & b^M_F \Sigma (b^M_F)' \end{pmatrix}, \qquad \mathrm{A^U} = \begin{pmatrix} \sigma_D^2 & 0 & 0 \\ 0 & \sigma_U^2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\Sigma$ is the diagonal matrix with $(\sigma_F^2, \sigma_\Theta^2, \sigma_D^2, \sigma_M^2, \sigma_U^2)$ as the elements. Note that the difference between the Kalman filtering of the two agents is that for U agents, the two error terms $v_t$ and $w_t$ are correlated. More specifically,

$$B \equiv E(vw') = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ b^M_{F3}\sigma_D^2 & b^M_{F5}\sigma_U^2 & 0 \end{pmatrix}.$$

Define

$$K^U = [(M^U \Omega^U (M^U)' + Q^U) H^U + B][(H^U)'(M^U \Omega^U (M^U)' + Q^U) H^U + A^U + (H^U)' B + B' H^U]^{-1},$$

where $\Omega^U$ satisfies:

$$\Omega^U = (M^U \Omega^U (M^U)' + Q^U) - K^U[(H^U)'(M^U \Omega^U (M^U)' + Q^U) + B'].$$

The updating of beliefs about state variables are:

$$\begin{aligned} \hat{\xi}^U_{t+1|t+1} &= M^U \hat{\xi}^U_{t|t} + K^U (y^U_{t+1} - (H^U)' M^U \hat{\xi}^U_{t|t}) \\ &= M^U \hat{\xi}^U_{t|t} + K^U (H^U)' M^U (\xi^U_t - \hat{\xi}^U_{t|t}) + K^U (w^U_{t+1} + (H^U)' v^U_{t+1}). \end{aligned}$$

This implies that (using $\xi^U_{t+1} = M^U \xi^U_t + v^U_{t+1}$):

$$\delta^U_{t+1} = (M^U - K^U (H^U)' M^U) \delta^U_t + K^U w^U_{t+1} - (1 - K^U (H^U)') v^U_{t+1},$$

where $\delta^U_t = \left(\delta^U_{F,t}, \delta^U_{\Theta,t}, \Delta^U_t\right)' \equiv \hat{\xi}^U_{t|t} - \xi^U_t$.

Recall that $\Pi^U = p_F F - p_\Theta \Theta + p_M \hat{F}^M = p_F \hat{F}^U - p_\Theta \widehat{\Theta}^U + p_M \bar{\bar{F}}$. Namely,

$$\delta^U_{\Theta,t} = \frac{p_F}{p_\Theta} \delta^U_{F,t} + \frac{p_M}{p_\Theta} \Delta^U.$$

Using this relationship, it follows that:

$$\begin{aligned}
\delta^U_{F,t+1} &= a^U_\delta \delta^U_{F,t} + c^U_\delta \Delta^U_t + b^U_\delta \epsilon_{t+1}, \\
\Delta^U_{F,t+1} &= a^U_\Delta \delta^U_{F,t} + c^U_\Delta \Delta^U_t + b^U_\Delta \epsilon_{t+1},
\end{aligned}$$

where $\epsilon_t = (\epsilon_{F,t}, \epsilon_{\Theta,t}, \epsilon_{D,t}, \epsilon_{M,t}, \epsilon_{U,t})$.

Having solved the belief updating process, we now solve the optimal portfolio choice of I, U and M agents given the assumed price format. The share excess demand is:

$$\begin{aligned}
Q_{t+1} &= P_{t+1} + D_{t+1} - RP_t \\
&= (p_M + p_{U2})(a^M_\delta - R)\delta^M_{F,t} + [p_{U1}(a^U_\delta - R) + p_{U2} a^U_\Delta]\delta^U_{F,t} \\
&\quad + [p_{U1} c^U_\delta + p_{U2}(c^U_\Delta - R)]\Delta^U_t - p_\Theta(a_\Theta - R)\Theta_t \\
&\quad + (p_F + p_M + p_{U1} + p_{U2} + 1)\epsilon_{F,t+1} + \epsilon_{D,t+1} - p_\Theta \epsilon_{\Theta,t+1} \\
&\quad + [(p_M + p_{U2}) b^M_\delta + p_{U1} b^U_\delta + p_{U2} b^U_\Delta]\epsilon_{t+1} \\
&= (p_M + p_{U2})(a^M_\delta - R)\delta^M_{F,t} + [p_{U1}(a^U_\delta - R) + p_{U2} a^U_\Delta]\delta^U_{F,t} \\
&\quad + [p_{U1} c^U_\delta + p_{U2}(c^U_\Delta - R)]\Delta^U_t - p_\Theta(a_\Theta - R)\Theta_t + b_Q \epsilon_{t+1}.
\end{aligned}$$

where

$$b_Q \equiv \big((p_{f0} + 1), -p_\Theta, 1, 0, 0\big) + [(p_M + p_{U2}) b^M_\delta + p_{U1} b^U_\delta + p_{U2} b^U_\Delta].$$

Thus the expectation of U, M agents are:

$$\begin{aligned}
E^M_t(Q_{t+1}) &= -p_\Theta(a_\Theta - R)\widehat{\Theta}^M_t + [p_{U1}(a^U_\delta - R) + p_{U2} a^U_\Delta](\delta^U_{F,t} - \delta^M_{F,t}) \\
&\quad + [p_{U1} c^U_\delta + p_{U2}(c^U_\Delta - R)]\Delta^U_t \\
&= -p_\Theta(a_\Theta - R)\Theta_t - [\, p_F(a_\Theta - R) + p_{U1}(a^U_\delta - R) + p_{U2} a^U_\Delta]\delta^M_{F,t} \\
&\quad + [p_{U1}(a^U_\delta - R) + p_{U2} a^U_\Delta]\delta^U_{F,t} + [p_{U1} c^U_\delta + p_{U2}(c^U_\Delta - R)]\Delta^U_t
\end{aligned}$$

$$\begin{aligned}
E^U_t(Q_{t+1}) &= -p_\Theta(a_\Theta - R)\widehat{\Theta}^U_t + (p_M + p_{U2})(a^M_\delta - R)\hat{\delta}^U_{F,t} \\
&= -p_\Theta(a_\Theta - R)\Theta_t + (p_M + p_{U2})(a^M_\delta - R)\delta^M_{F,t}
\end{aligned}$$

$$-[\, p_F(a_\Theta - R) + (p_M + p_{U2})(a^M_\delta - R)]\delta^U_{F,t} + [(p_M + p_{U2})(a^M_\delta - R) - p_M(a_\Theta - R)]\Delta^U_t$$

Here in the second equations of both expectations we use the substitution of $\widehat{\Theta}^M_t$, $\widehat{\Theta}^U_t$, $\hat{\delta}^U_{F,t}$ to make the expression as format of pricing function. We will use this in the market clearing condition to solve for the coefficients in the price function.

Thus the forecasting errors of agents M and U on $Q_{t+1}$ are (using the fact that $\hat{\delta}^U_{F,t} - \delta^M_{F,t} = \Delta^U_t - \delta^U_{F,t}$ ):

$$E^M_t(Q_{t+1}) - Q_{t+1} = \left[-(p_M + p_{U2})(a^M_\delta - R) - (p_{U1}(a^U_\delta - R) + p_{U2}a^U_\Delta)\right]\delta^M_{F,t} - p_\Theta(a_\Theta - R)\delta^M_{\Theta,t} - b_Q\epsilon_{t+1}$$

$$\begin{aligned} E^U_t(Q_{t+1}) - Q_{t+1} = &\left[-(p_M + p_{U2})(a^M_\delta - R) - \left[p_{U1}(a^U_\delta - R) + p_{U2}a^U_\Delta\right]\right]\delta^U_{F,t} - p_\Theta(a_\Theta - R)\delta^U_{\Theta,t} \\ &+ \left[(p_M + p_{U2})(a^M_\delta - R) - [p_{U1}c^U_\delta + p_{U2}(c^U_\Delta - R)]\right]\Delta^U_t - b_Q\epsilon_{t+1}. \end{aligned}$$

Define the following vectors:

$$a^M_Q = (-(p_M + p_{U2})(a^M_\delta - R) - (p_{U1}(a^U_\delta - R) + p_{U2}a^U_\Delta), -p_\Theta(a_\Theta - R))$$

$$\begin{aligned} a^U_Q = \big(&-(p_M + p_{U2})(a^M_\delta - R) - \left[p_{U1}(a^U_\delta - R) + p_{U2}a^U_\Delta\right], \\ &-p_\Theta(a_\Theta - R), (p_M + p_{U2})(a^M_\delta - R) - \left[p_{U1}c^U_\delta + p_{U2}(c^U_\Delta - R)\right]\big). \end{aligned}$$

Then the conditional variances are:

$$var_{M,t}(Q_{t+1}) = a^M_Q\Omega^M(a^M_Q)' + b_Q\Sigma b_Q{}'$$

$$var_{U,t}(Q_{t+1}) = a^U_Q\Omega^U(a^U_Q)' + b_Q\Sigma b_Q{}'.$$

It follows that the demand of the two types of agents are:

$$\theta^I = g^I_M\delta^M_F + g^I_{U1}\delta^U_F + g^I_{U2}\Delta^U + g^I_\Theta\Theta$$

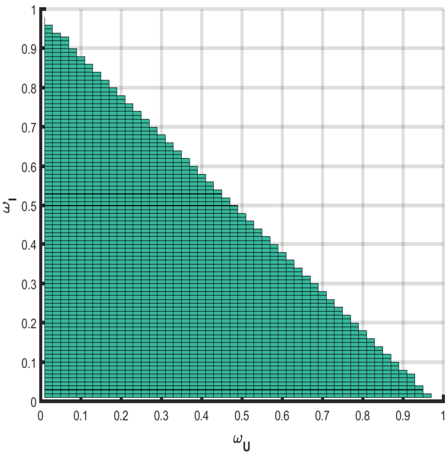
$$\theta^U = g^U_M\delta^M_F + g^U_{U1}\delta^U_F + g^U_{U2}\Delta^U + g^U_\Theta\Theta,$$

$$\theta^M = g^M_M\delta^M_F + g^M_{U1}\delta^U_F + g^M_{U2}\Delta^U + g^M_\Theta\Theta.$$

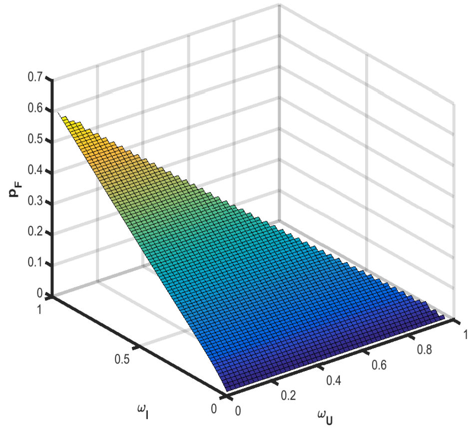
Market clearing condition states $\omega_I\theta^I + \omega_U\theta^U + \omega_M\theta^M = \Theta$ . The matching of the coefficients will solve the pricing function.

Q.E.D.

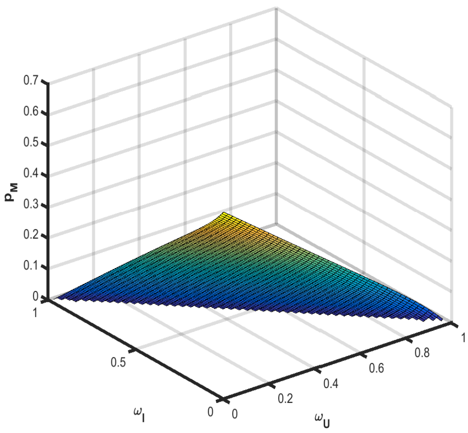
## Figure 1. Price Function



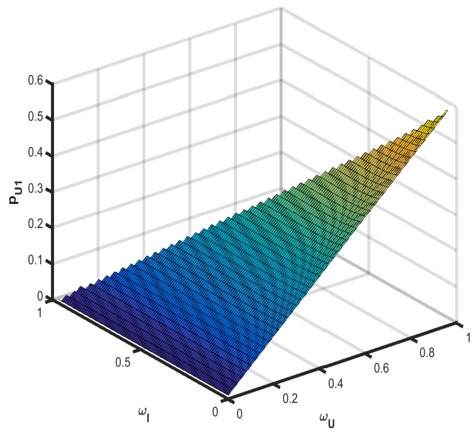

Panel A. Base



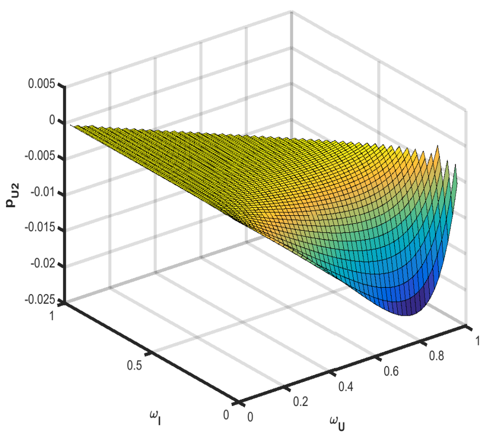

Panel B. $p_F$



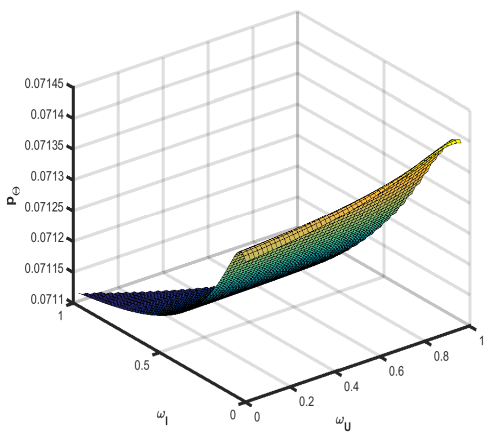

Panel C: $p_M$




Panel D: $p_{U1}$



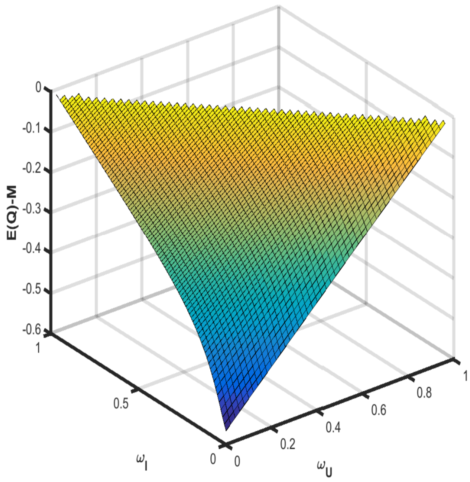

Panel E: $p_{U2}$



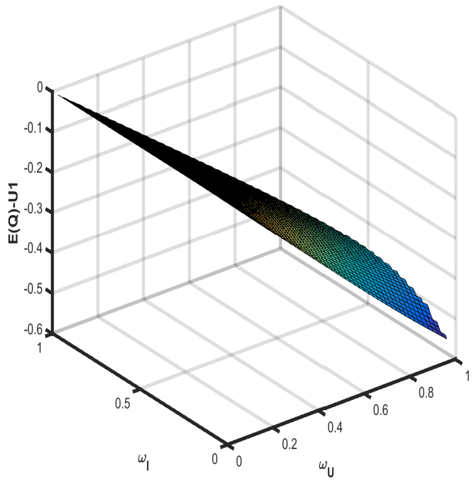

Panel F: $p_\Theta$

**Figure 2. Expected Excess Share Returns: Informed Agents I**

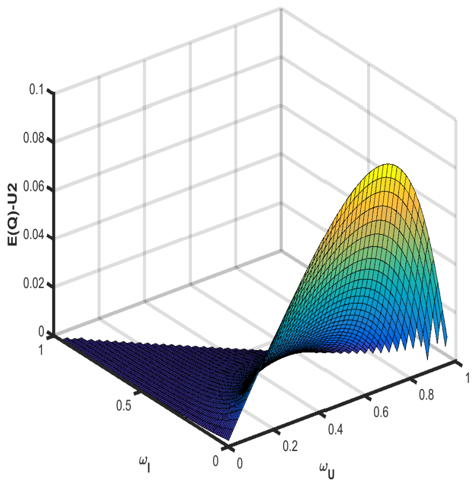
This figure shows the expected excess share return from the informed agents I's point of view: $E_t^I(Q_{t+1})$. It presents the coefficients of the four state variables. The coefficient of $F_t$ is always zero.



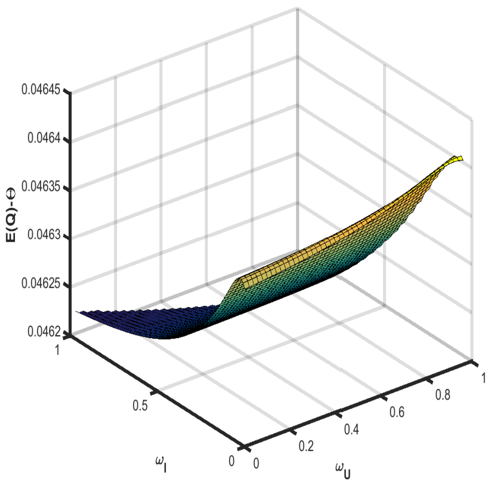

$\delta_F^M$




$\delta_F^U$



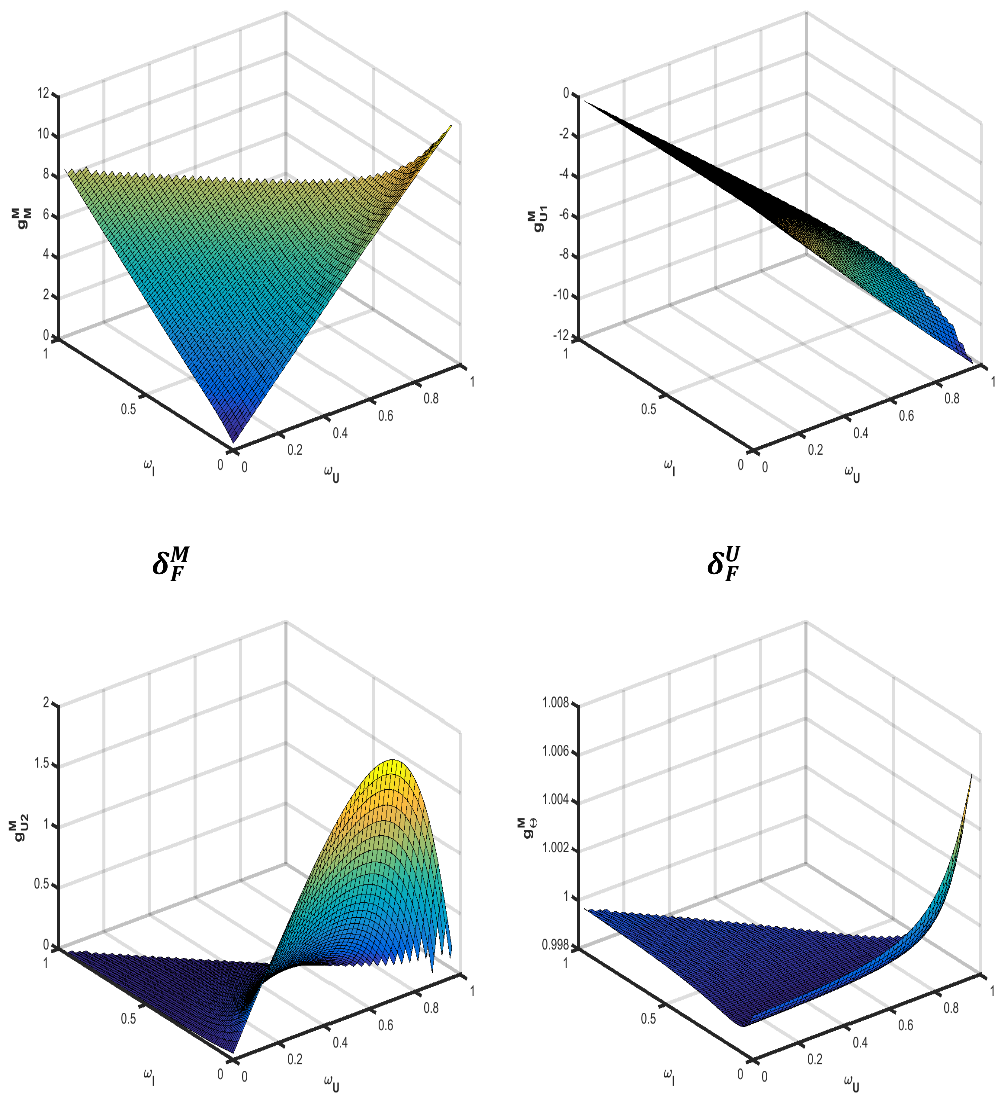

$\Delta$




$\Theta$

**Figure 3 Individual Demand: Partially Informed Agents M**

This figure shows the individual stock demand of the partially informed agents M. It presents the coefficients of the four state variables known to I. The coefficient of $F_t$ is always zero. Note that M's state variables in the expected excess returns are different. They are their own beliefs of the fundamentals $(F, \Theta)$ and uninformed agents U's first and second order beliefs.

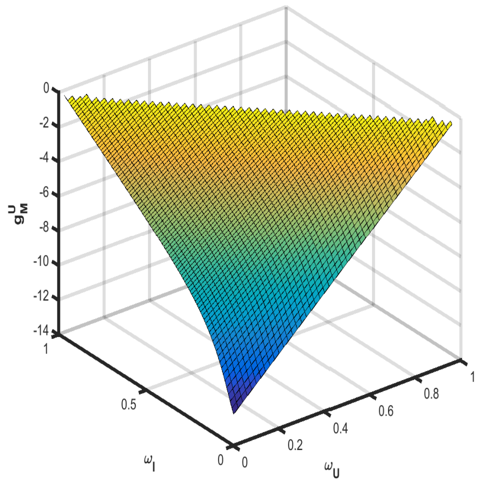



$\boldsymbol{\delta_F^M}$ $\boldsymbol{\delta_F^U}$

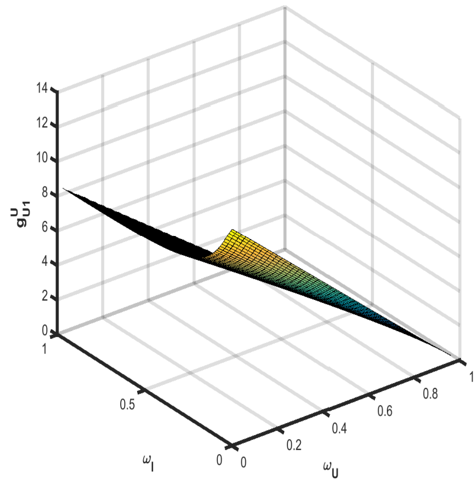
$\boldsymbol{\Delta}$ $\boldsymbol{\Theta}$

**Figure 4. Individual Demand: Uninformed Agents U**

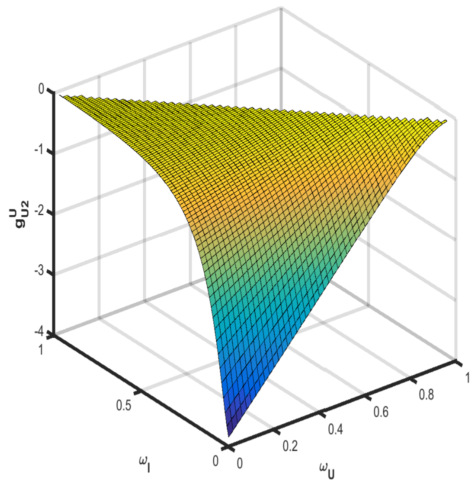
This figure shows the individual stock demand of the uninformed agents U. It presents the coefficients of the four state variables known to I. The coefficient of $F_t$ is always zero. Note that U's state variables in the expected excess returns are different. They are their own beliefs of the fundamentals $(F, \Theta)$ and their beliefs about M's expectation error.



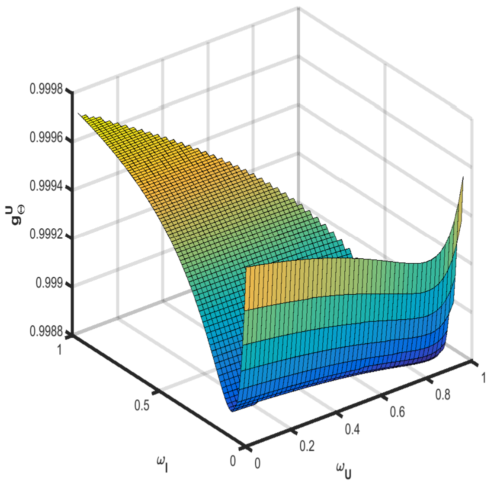

$\boldsymbol{\delta_F^M}$




$\boldsymbol{\delta_F^U}$

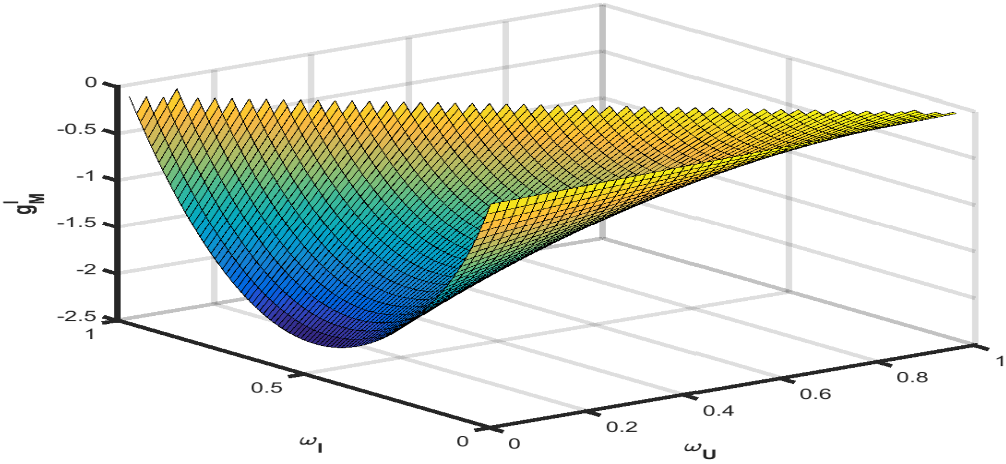



$\boldsymbol{\Delta}$

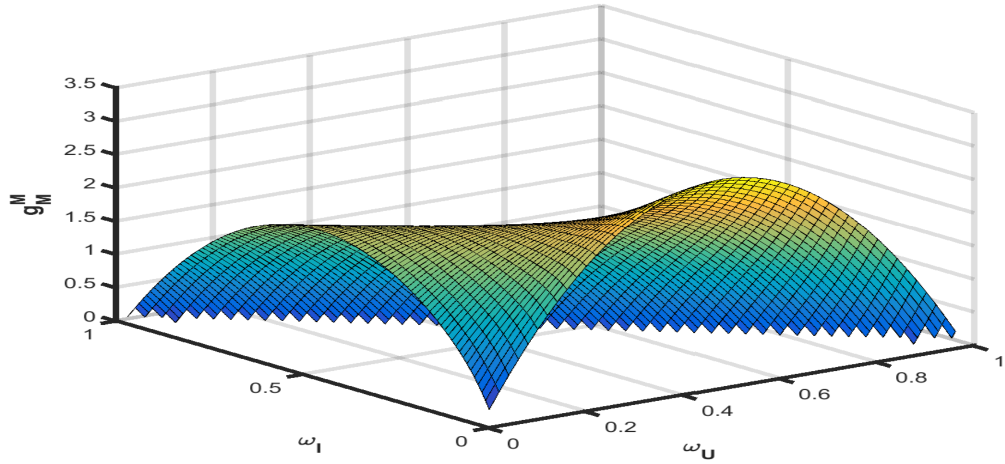



$\boldsymbol{\Theta}$

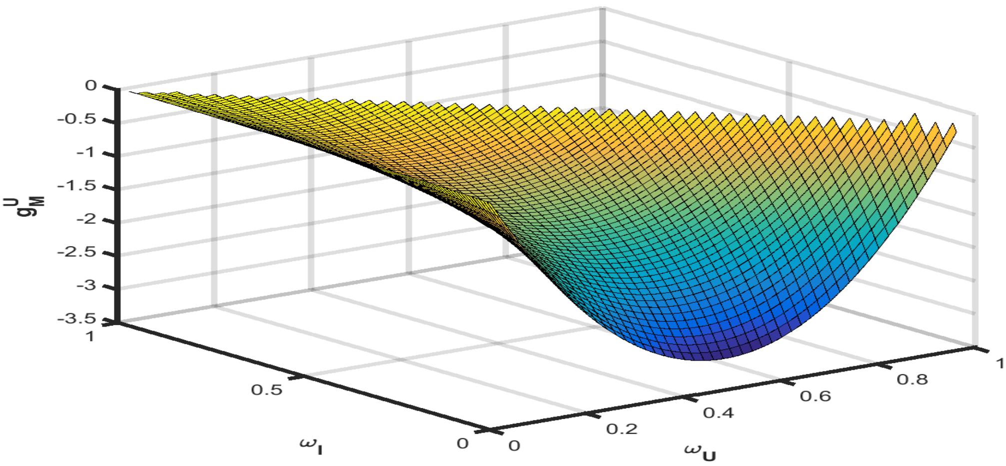
**Figure 5a. Group Demand (Trading): $\delta_F^M$ Coefficient**




Panel A. Informed Agents I

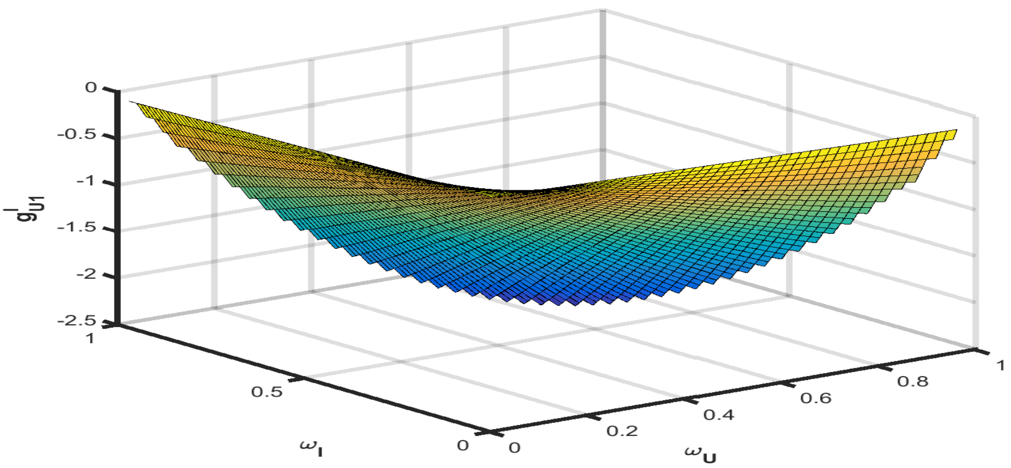



Panel B. Partially Informed Agents M

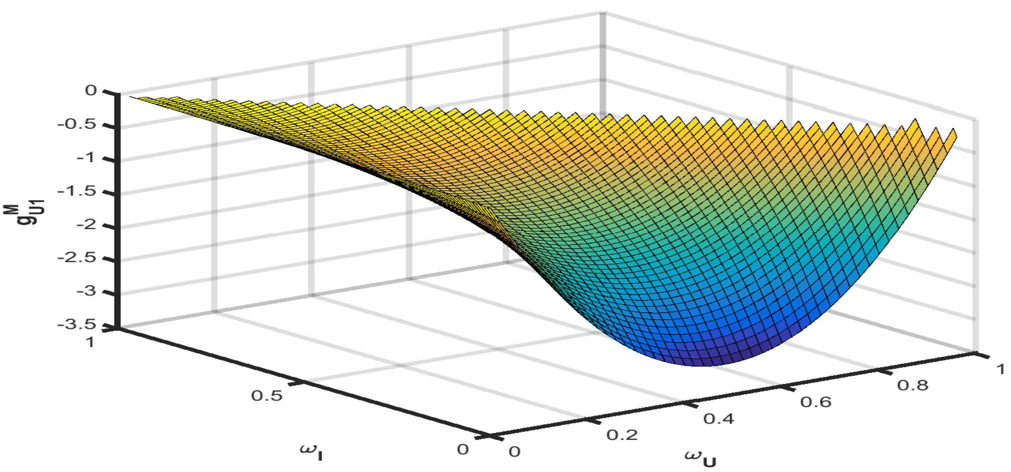



Panel C. Uninformed Agents U

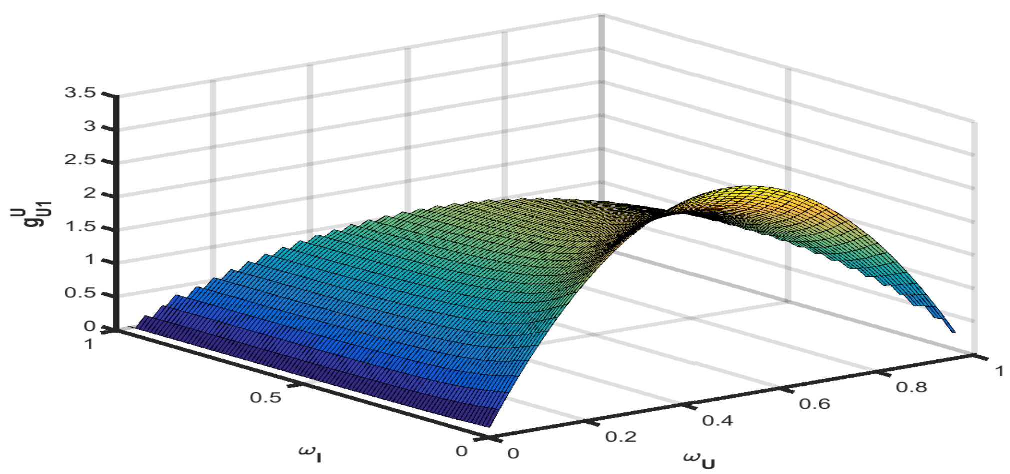
**Figure 5b. Group Demand (Trading): $\delta_F^U$ Coefficient**




Panel A. Informed Agents I

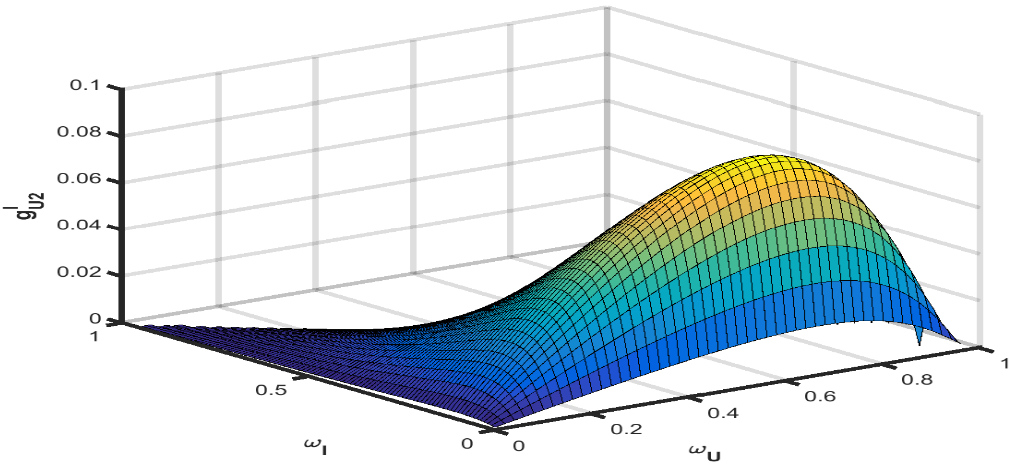



Panel B. Partially Informed Agents M

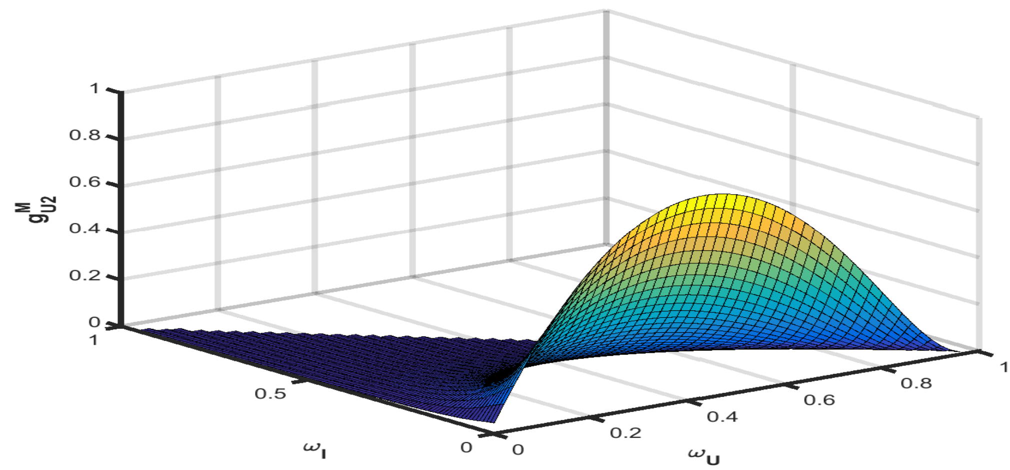



Panel C. Uninformed Agents U

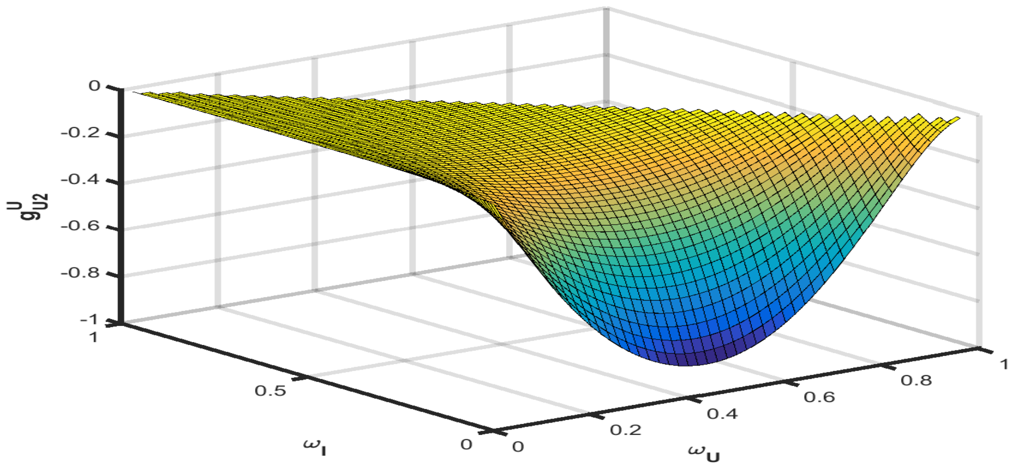
**Figure 5c. Group Demand (Trading): Δ Coefficient**




Panel A. Informed Agents I

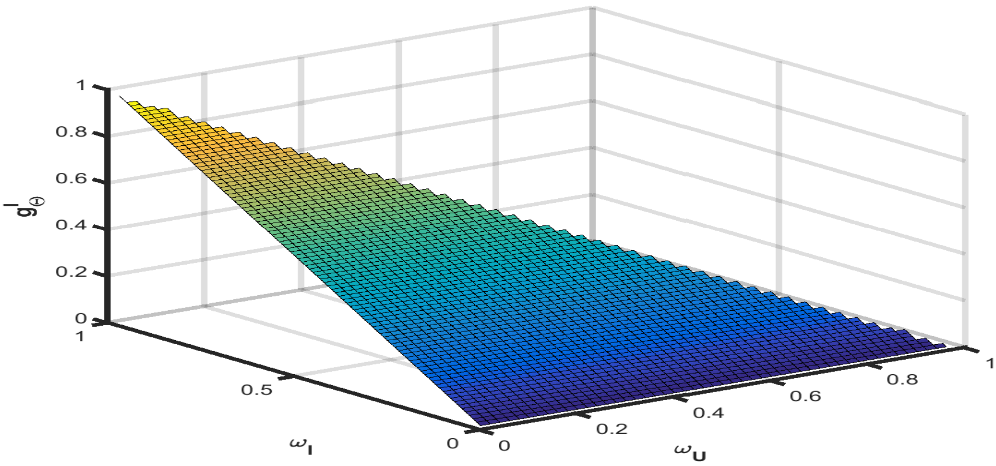



Panel B. Partially Informed Agents M

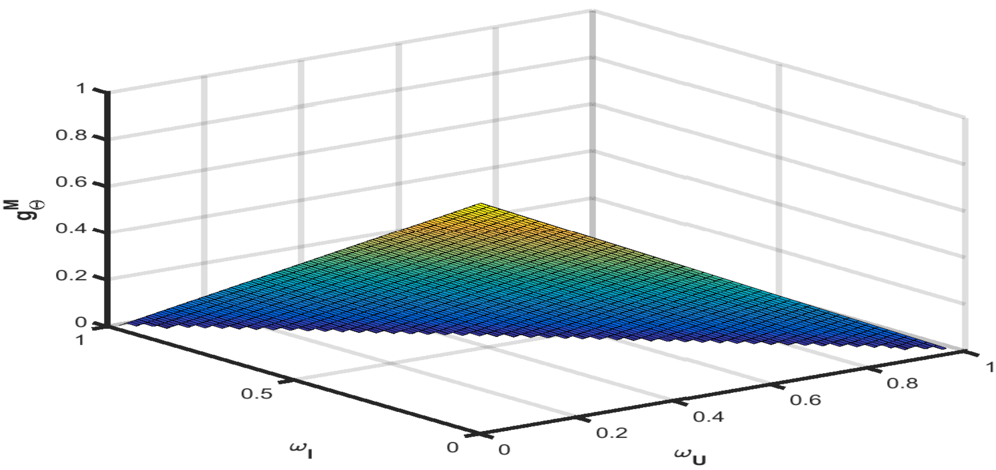



Panel C. Uninformed Agents U.

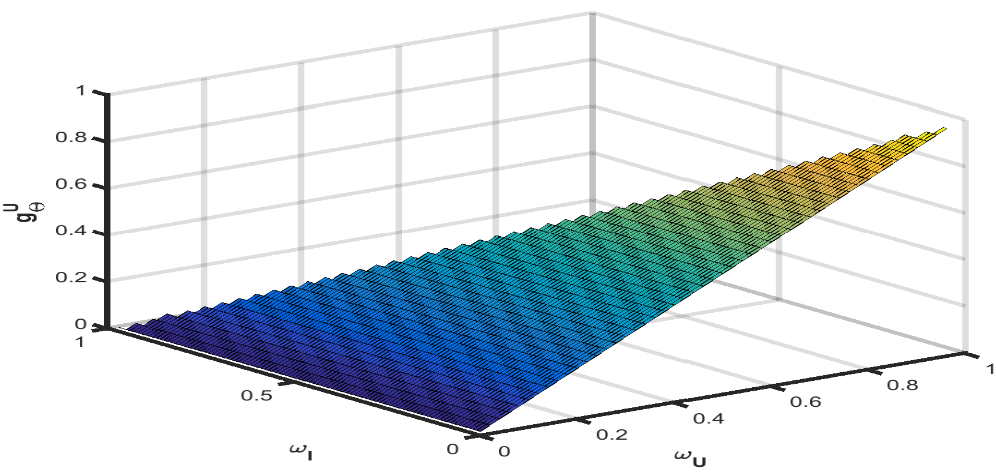
**Figure 5d. Group Demand (Trading): Θ Coefficient**




Panel A. Informed Agents I




Panel B. Partially Informed Agents M




Panel C. Uninformed Agents U